Basic Sets

Example 1. Let $S = \{1, \{2, 3\}, 4\}$. Indicate whether each statement is true or false.

(a) $|S| = 4$

Solution: False. Note that the elements of S are 1, the set $\{2, 3\}$, and 4. Thus $|S| = 3$.

(b) $\{1\} \in S$

Solution: False. While $1 \in S$, the set $\{1\}$ is not an element of S.

(c) $\{2, 3\} \in S$

Solution: True

(d) $\{1, 4\} \subseteq S$

Solution: True

(e) $2 \in S$.

Solution: False. As discussed in part (a), the elements of S are 1, the set $\{2, 3\}$, and 4, thus 2 is not an element of S.

(f) $S = \{1, 4, \{2, 3\}\}$

Solution: True

(g) $\emptyset \subseteq S$

Solution: True

Example 2. Compute the cardinality of the set, E, where E is defined as

$$E = \{x \in \mathbb{R} \ : \ \sin(x) = 1/2 \text{ and } |x| \leq 5\}$$

Solution: $\sin x = 1/2$ when $x = \pi/6 + 2k\pi$ for $k \in \mathbb{Z}$ or when $x = 5\pi/6 + 2k\pi$ for $k \in \mathbb{Z}$. The only ones that also satisfy $|x| \leq 5$ are $x = -7\pi/6, \pi/6, 5\pi/6$. So E can be written as $E = \{-7\pi/6, \pi/6, 5\pi/6\}$ which has cardinality 3.

Example 3. Suppose $A = \{0, 2, 4, 6, 8\}, B = \{1, 3, 5, 7\}$ and $C = \{2, 8, 4\}$. Find:
(a) $A \cup B$ (b) $A \setminus C$ (c) $B \setminus A$ (d) $B \cap C$ (e) $C \setminus B$

Solution: (a) $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

(b) $A \setminus C = \{0, 6\}$

(c) $B \setminus A = B$

(d) $B \cap C = \emptyset$

(e) $C \setminus B = C$

Example 4. Prove that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Solution: Let $a \in \{9^n : n \in \mathbb{Z}\}$. Then there exists $k \in \mathbb{Z}$ such that $a = 9^k$. Note that $a = 3^{2k}$ and $2k \in \mathbb{Z}$ since the integers are closed under multiplication. Thus $a \in \{3^n : n \in \mathbb{Z}\}$. We have proven that $a \in \{9^n : n \in \mathbb{Z}\}$ implies $a \in \{3^n : n \in \mathbb{Z}\}$, i.e., $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$.
Now, consider $3 \in \{3^n : n \in \mathbb{Z}\}$. Note that $3 = 3^1 = 9^{\frac{1}{2}}$ and 3 cannot be written as an integer power of 9. Thus, $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Example 5. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Solution: Let $a \in \{9^n : n \in \mathbb{Q}\}$. Then there exists $p \in \mathbb{Q}$ such that $a = 9^p$. Note that $a = 3^{2p}$ and $2p \in \mathbb{Q}$ since the rationals are closed under multiplication. Thus $a \in \{3^n : n \in \mathbb{Q}\}$. We have proven that $a \in \{9^n : n \in \mathbb{Q}\}$ implies $a \in \{3^n : n \in \mathbb{Q}\}$, i.e., $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.
On the other hand, let $a \in \{3^n : n \in \mathbb{Q}\}$. Then there exists $q \in \mathbb{Q}$ such that $a = 3^q$. Note that $a = 9^{\frac{q}{2}}$ and $\frac{q}{2} \in \mathbb{Q}$. Thus $a \in \{9^n : n \in \mathbb{Q}\}$. We have proven that $a \in \{3^n : n \in \mathbb{Q}\}$ implies $a \in \{3^n : n \in \mathbb{Q}\}$, i.e., $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.
Thus, $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Functions

Example 6. For each of the following, determine the largest set $A \subseteq \mathbb{R}$, such that $f : A \to \mathbb{R}$ defines a function. Next, determine the range, $f(A) := \{y \in \mathbb{R} : f(x) = y, \text{ for some } x \in A\}$.

(a) $f(x) = 1 + x^2$,

(b) $f(x) = 1 - \frac{1}{x}$,

(c) $f(x) = \sqrt{3x-1}$,

(d) $f(x) = x^3 - 8$,

(e) $f(x) = \frac{x}{x-3}$.

Solution:

(a) Any element in $\mathbb{R}$ has a unique image in $\mathbb{R}$ under $f(x) = 1 + x^2$, thus $A = \mathbb{R}$. We claim that $f(A) = [1, \infty) = \{y \in \mathbb{R} : y \geq 1\}$. First we need to show that $f(A) \subseteq [1, \infty)$. Indeed, for any $x \in \mathbb{R}$, $1 + x^2 \in \mathbb{R}$ and $1 + x^2 \geq 1$, thus $f(x) \in [1, \infty)$. Now, take an arbitrary element $y \in [1, \infty)$. Define $x = \sqrt{y-1} \in \mathbb{R}$. Then $f(x) = y$, i.e., $y \in f(A)$, implying that $[1, \infty) \subseteq f(A)$.

(b) Any element in $\mathbb{R}$ but 0 has a unique image in $\mathbb{R}$ under f, thus $A = \mathbb{R} \setminus \{0\}$. We claim that $f(A) = \mathbb{R} \setminus \{1\}$. First we need to show that $f(A) \subseteq \mathbb{R} \setminus \{1\}$. Indeed, for any $x \in \mathbb{R} \setminus \{0\}$, $1 + \frac{1}{x} \in \mathbb{R} \setminus \{1\}$, since it is clearly a real number and $\frac{1}{x} \neq 0$ for all x in the domain. Now, take an arbitrary element $y \in \mathbb{R} \setminus \{1\}$. Define $x = \frac{1}{y-1} \in \mathbb{R} \setminus \{0\}$. Then $f(x) = y$, i.e., $y \in f(A)$, implying that $\mathbb{R} \setminus \{1\} \subseteq f(A)$.

(c) Any real number greater than or equal to $\frac{1}{3}$ has a unique image in $\mathbb{R}$ under f. Also, no real number less than $\frac{1}{3}$ is mapped to a real number under f, so $A = [\frac{1}{3}, \infty)$. We claim that $f(A) = [0, \infty)$. First we need to show that $f(A) \subseteq [0, \infty)$. Indeed, for any $x \in [\frac{1}{3}, \infty)$, $\sqrt{3x-1} \in [0, \infty)$, thus $f(x) \in [0, \infty)$. Now, take an arbitrary element $y \in [0, \infty)$. Define $x = \frac{y^2+1}{3} \in [\frac{1}{3}, \infty)$. Then $f(x) = y$, i.e., $y \in f(A)$, implying that $[0, \infty) \subseteq f(A)$.

(d) Any element in $\mathbb{R}$ has a unique image in $\mathbb{R}$ under $f(x) = x^3 - 8$, thus $A = \mathbb{R}$. We claim that $f(A) = \mathbb{R}$. It is clear $f(A) \subseteq \mathbb{R}$, since f maps real numbers to real numbers. It remains to show $\mathbb{R} \subseteq f(A)$. Let $y \in \mathbb{R}$ be arbitrary. Define $x = \sqrt[3]{y+8}$. Note that x so defined is in the domain of f and $f(x) = y$. Thus, $y \in f(A)$, i.e., $\mathbb{R} \subseteq f(A)$. Thus we can conclude $\mathbb{R} = f(A)$.

(e) Any element in $\mathbb{R}$ but 3 has a unique image in $\mathbb{R}$ under f, thus $A = \mathbb{R} \setminus \{3\}$. We claim that $f(A) = \mathbb{R} \setminus \{1\}$. First we need to show that $f(A) \subseteq \mathbb{R} \setminus \{1\}$. Indeed, for any $x \in \mathbb{R} \setminus \{3\}$, $1 + \frac{3}{x-3} \in \mathbb{R} \setminus \{1\}$, since it is clearly a real number and $\frac{3}{x-3} \neq 0$ for all x in the domain. Now, take an arbitrary element $y \in \mathbb{R} \setminus \{1\}$. Define $x = \frac{3}{y-1} + 3 \in \mathbb{R} \setminus \{3\}$. Then $f(x) = y$, i.e., $y \in f(A)$, implying that $\mathbb{R} \setminus \{1\} \subseteq f(A)$. Thus we can conclude $f(A) = \mathbb{R} \setminus \{1\}$.

Injective, Surjective, Bijective Functions

Example 7. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((m, n)) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.

Solution: This function is not injective since $f(0, 0) = 0$ and $f(1, 2) = 0$ and yet $(0, 0) \neq (1, 2)$. The function is also not surjective because there are no values $n, m \in \mathbb{Z}$ so that $f(m, n) = 3$. This is because $2(n - 2m) \neq 3$ for any $n, m \in \mathbb{Z}$ since 3 is not even.

Example 8. Define the operation

$$f(p) := \frac{d}{dx}p.$$

Does f define a function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f an injective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f a surjective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer.

Solution: Yes f is a function. For each input there is a unique output.
No f is not injective. $f(1) = f(2) = 0$ and yet $1 \neq 2$.
No f is not surjective. There is no polynomial p so that $f(p) = x^4$ (it would have to be degree 5 which is not in the domain).

Example 9. Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

Solution: **Showing that f is injective:** Take $x_1, x_2 \in \mathbb{R} \setminus \{2\}$ and $f(x_1) = f(x_2)$ then:

$$\begin{aligned}
f(x_1) &= f(x_2) \\
\frac{5x_1 - 1}{x_1 - 2} &= \frac{5x_2 - 1}{x_2 - 2} \\
(5x_1 - 1)(x_2 - 2) &= (5x_2 - 1)(x_1 - 2) \\
5x_1x_2 - x_2 - 10x_1 + 2 &= 5x_2x_1 - x_1 - 10x_2 + 2 \\
-x_2 - 10x_1 &= -x_1 - 10x_2 \\
-9x_1 &= -9x_2 \\
x_1 &= x_2
\end{aligned}$$

So by definition f is injective.

Showing that f is surjective: Take $y \in \mathbb{R} \setminus \{5\}$ then:

Scratch Work:

$$\begin{aligned}
\frac{5x + 1}{x - 2} &= y \\
5x + 1 &= y(x - 2) \\
5x + 1 &= yx - 2y \\
5x - yx &= -2y - 1 \\
x(5 - y) &= -2y - 1 \\
x &= \frac{-2y - 1}{5 - y}
\end{aligned}$$

So Take $x = \dfrac{-2y-1}{5-y}$ (Note: x is well defined since $y \neq 5$. Also notice that $x \neq 2$ so $x \in \mathbb{R} \setminus \{2\}$ as desired

). Then we have:

$$\begin{aligned} f(x) &= f\left(\frac{-2y-1}{5-y}\right) \\ &= \frac{5\left(\frac{-2y-1}{5-y}\right)+1}{\left(\frac{-2y-1}{5-y}\right)-2} \\ &= \frac{5(-2y-1)+(5-y)}{(-2y-1)-2(5-y)} \\ &= \frac{-10y-5+5-y}{-2y-1-10+2y} \\ &= \frac{-11y}{-11} = y \end{aligned}$$

So by definition f is surjective.

Since f is both surjective and injective it is therefore bijective.

Example 10. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$ is injective.
Hint: A graph can help, but a graph is not a proof.

Solution: Looking at the graph we can see that $f(0) = 0$ and $f(1) = 0$ since $0 \neq 1$ and yet $f(0) = f(1)$ we know that f is not injective!

Example 11. Let $A = \mathbb{R} \setminus \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$.

(i) Prove that f is bijective.

(ii) Determine f^{-1}.

Solution:

(i) **Injective:** take $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ then:

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1}{x_1-1} &= \frac{x_2}{x_2-1} \\ x_1(x_2-1) &= x_2(x_1-1) \\ x_1x_2 - x_1 &= x_2x_1 - x_2 \\ -x_1 &= -x_2 \\ x_1 &= x_2 \end{aligned}$$

so f is injective.

Surjective: take $y \in A$

Scratch Work:

$$\begin{aligned} \frac{x}{x-1} &= y \\ x &= y(x-1) \\ x &= yx - y \\ x - xy &= -y \\ x(1-y) &= -y \\ x &= \frac{-y}{1-y} \end{aligned}$$

and let $x = \dfrac{-y}{1-y}$ then we can see that:

$$\begin{aligned} f(x) &= f\left(\frac{-y}{1-y}\right) \\ &= \frac{\left(\frac{-y}{1-y}\right)}{\left(\frac{-y}{1-y}\right) - 1} \\ &= \frac{-y}{-y-(1-y)} \\ &= \frac{-y}{-1} = y \end{aligned}$$

and so f is surjective. Since f is injective and surjective it is bijective.

(ii) Using the above scratch work we see that $f^{-1}(x) = \dfrac{-x}{1-x}$. The work for surjective shows that $f(f^{-1}(x)) = x$. I will leave it for the students to show that $f^{-1}(f(x)) = x$.

Example 12. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(x, y) = (2x - 3y, x + 1)$$

(a) Show that f is a bijection.

(b) Determine the inverse f^{-1} of f.

Solution:

(a) **Injective:** Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and that $f(x_1, y_1) = f(x_2, y_2)$ then:

$$\begin{aligned}
f(x_1, y_1) &= f(x_2, y_2) \\
(2x_1 - 3y_1, x_1 + 1) &= (2x_2 - 3y_2, x_2 + 1) \\
x_1 + 1 &= x_2 + 1 && \text{(Looking at the 2nd coordinate)} \\
x_1 &= x_2 && (\star) \\
2x_1 - 3y_1 &= 2x_2 - 3y_2 && \text{(Looking at the 1st coordinate)} \\
2x_1 - 3y_1 &= 2x_1 - 3y_2 && \text{(by } (\star)\text{)} \\
-3y_1 &= -3y_2 \\
y_1 &= y_2
\end{aligned}$$

Therefore $(x_1, y_1) = (x_2, y_2)$ so f is injective.

Surjective: Take $(u, v) \in \mathbb{R}^2$. Let's try to find $(x, y) \in \mathbb{R}^2$ so that $f(x, y) = (u, v)$. So we take $x = v - 1$ and $y = \dfrac{u - 2(v-1)}{-3}$ and check that:

$$\begin{aligned}
f(x, y) &= f\left(v - 1, \frac{u - 2(v-1)}{-3}\right) \\
&= \left(2(v-1) - 3\left(\frac{u - 2(v-1)}{-3}\right), (v-1) + 1\right) \\
&= (2(v-1) + (u - 2(v-1)), v) \\
&= (2v - 2 + u - 2v + 2, v) \\
&= (u, v)
\end{aligned}$$

and so f is surjective. Since f is injective and surjective it is bijective.

(b) Using the scratch work above we know $f^{-1}(x, y) = \left(x - 1, \dfrac{x - 2(y-1)}{-3}\right)$.

The above work shows that $f(f^{-1}(x, y)) = (x, y)$. One can also show that $f^{-1}(f(x, y)) = (x, y)$.

Example 13. Define the function, $f : \mathbb{P}_3 \to \mathbb{R}$ via the operation

$$f(p) := \int_0^1 p(x)dx.$$

Is f injective and or surjective from $\mathbb{P}_3$ to $\mathbb{R}$? Justify your answer.

Solution: The above function is not injective. Consider the two polynomials $p, q \in \mathbb{P}_3$ defined by $p(x) = 1$ and $q(x) = 2x$, for $x \in \mathbb{R}$. Then $f(p) = 1$ and $f(q) = x^2|_0^1 = 1$. Thus, $p \neq q$, but $f(p) = f(q)$, i.e. f is not injective.
On the other hand, f is surjective. Indeed, let $r \in \mathbb{R}$ be arbitrary and define the constant polynomial $s(x) = r\, \forall x \in \mathbb{R}$. Note tat $s \in \mathbb{P}_3$ and $f(s) = r$. Thus, for every $r \in \mathbb{R}$ there exists a polynomial $s \in \mathbb{P}_3$ such that $f(s) = r$, so f is surjective.

Example 14. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse. Carefully justify that your answer does indeed yield the inverse function.

General Math Proofs

Example 15. Assume that you know that $x < y$. Carefully justify the statement that

$$x < \frac{x+y}{2} < y.$$

Solution: The conclusion here is equivalent to $x < \dfrac{x+y}{2}$ and $\dfrac{x+y}{2} < y$. We will try to prove these two individually:

$$\begin{aligned} x &< y \\ 2x &< x + y && \text{(add } x \text{ to both sides)} \\ x &< \frac{x+y}{2} && \text{(divide by 2 } (\star)) \\ x &< y \\ x + y &< 2y && \text{(add } y \text{ to both sides)} \\ \frac{x+y}{2} &< y && \text{(divide by 2 } (\star\star)) \end{aligned}$$

$(\star)$ and $(\star\star)$ together give us the desired result:

$$x < \frac{x+y}{2} < y.$$

Example 16. Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Solution: Assume, by way of contradiction, that $a, b \in \mathbb{R}$ and a is rational, ab is irrational, and b is rational. Then, there exist $p, q, r, s \in \mathbb{Z}$, $q \neq 0$ and $s \neq 0$, such that $a = \frac{p}{q}$ and $b = \frac{r}{s}$. Therefore, $ab = \frac{pr}{qs}$ and we can conclude $ab \in \mathbb{Q}$, as $pr, qs \in \mathbb{Z}$ with $qs \neq 0$. This contradicts the assumption that ab is irrational. Thus the assumption that b is rational cannot be true, therefore b must be irrational.

Example 17. Show that there exists a positive even integer m such that for every positive integer n,

$$\left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{2}.$$

Solution: Consider $m = 2$ and lets consider various cases for n.

If $n = 1$ then

$$\left|\frac{1}{2} - \frac{1}{1}\right| \leq \frac{1}{2}$$
$$\left|-\frac{1}{2}\right| \leq \frac{1}{2} \qquad \text{(true)}$$

If $n \geq 2$ then $\frac{1}{n} \leq \frac{1}{2}$ and $\frac{1}{2} \leq \frac{1}{2} + \frac{1}{n}$ then we know

$$\frac{1}{n} \leq \frac{1}{2} \leq \frac{1}{2} + \frac{1}{n} \qquad \text{(subtract } 1/n \text{ on all sides)}$$
$$0 \leq \frac{1}{2} - \frac{1}{n} \leq \frac{1}{2} \qquad \text{(if } A \geq 0 \text{ then } A = |A|)$$
$$0 \leq \left|\frac{1}{2} - \frac{1}{n}\right| \leq \frac{1}{2}$$

Example 18. Prove: For every real number $x \in [0, \pi/2]$, we have $\sin x + \cos x \geq 1$.

Solution: Consider a proof by contradiction. Suppose there is a real number $x \in [0, \pi/2]$ so that $\sin x + \cos x < 1$.

Since $x \in [0, \pi/2]$, neither $\sin x$ nor $\cos x$ is negative, so $0 \leq \sin x + \cos x < 1$. Thus $0^2 \leq (\sin x + \cos x)^2 < 1^2$, which gives $0^2 \leq \sin^2 x + 2\sin x \cos x + \cos^2 x < 1^2$. As $\sin^2 x + \cos^2 x = 1$, this becomes $0 \leq 1 + 2\sin x \cos x < 1$, so $1 + 2\sin x \cos x < 1$. Subtracting 1 from both sides gives $2\sin x \cos x < 0$. But this contradicts the fact that neither $\sin x$ nor $\cos x$ is negative.

Example 19. Suppose $x, y \in \mathbb{R}^+$. Prove if $xy > 100$ then $x > 10$ or $y > 10$.

Logic

Example 20. For the sets $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P : A \subseteq B. \qquad Q : |A \setminus B| = 6.$$

Determine which of the following statements are true.
(a) $P \vee Q$ (b) $P \vee (\neg Q)$ (c) $P \wedge Q$ (d) $(\neg P) \wedge Q$ (e) $(\neg P) \vee (\neg Q)$.

Solution:

Scratch Work: First lets determine the truth values of P and Q.
P is false since $3 \in A$ and yet $3 \notin B$.
Q is true since $A \setminus B = \{1, 3, 5, 7, 8, 10\}$

(a) True

(b) False

(c) False

(d) True

(e) True

Example 21. Let P : 15 is odd. and Q : 21 is prime. State each of the following in words, and determine whether they are true or false.
(a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\neg P) \vee Q$ (d) $P \wedge (\neg Q)$

Solution:

(a) 15 is odd or 21 is prime. **True**.

(b) 15 is odd and 21 is prime. **False**.

(c) 15 is not odd or 21 is prime. **False**.

(d) 15 is odd and 21 is not prime. **True**.

Example 22. Rewrite the following using logical connectives and quantifiers

(a) If f is a polynomial and its degree is greater than 2, then f' is not constant.

(b) The number x is positive but the number y is not positive.

Solution:

(a) (f is a polynomial such that $\deg(f) \geq 2$) $\Longrightarrow$ ($f' \not\equiv$ const)

(b) $x > 0 \wedge \neg(y > 0)$

Example 23. Which of the following best identifies $f : \mathbb{R} \to \mathbb{R}$ as a constant function, where x and y are real numbers.

(a) $\exists x, \forall y, f(x) = y$.

(b) $\forall x, \exists y, f(x) = y$.

(c) $\exists y, \forall x, f(x) = y$.

(d) $\forall y, \exists x,\ f(x) = y$.

Solution: Statement (c) defines f as a constant function. It states that there is an element y such that all the input values x get mapped to it.

Example 24. Negate the following statements:

(a) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z},\ x + y = 1$.

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z},\ xy = x$.

Solution: The negations of the above statements are as follows.

(a) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z},\ x + y \neq 1$.

(b) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z},\ xy \neq x$.

Example 25. In each of the following cases explain what is meant by the statement and decide whether it is true or false.

(a) $\lim\limits_{x\to c} f(x) = L$ if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

(b) $\lim\limits_{x\to c} f(x) = L$ if $\exists \delta > 0 \quad \forall \varepsilon > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

(c) $f : A \to B$ is surjective provided $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.

Solution:

(a) The statement "$\lim\limits_{x\to c} f(x) = L$ if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$." is **true**. This is the definition of $\lim\limits_{x\to c} f(x) = L$.

(b) The statement "$\lim\limits_{x\to c} f(x) = L$ if $\exists \delta > 0 \quad \forall \varepsilon > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$." is **false**. This requires that the same value of δ works for all values of ε, which is not the case, unless the function is constant.

(c) The statement "$f : A \to B$ is surjective provided $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$." is **true**. This is the definition of surjectivity.

Induction

Example 26. If n is a non-negative integer, use mathematical induction to show that $5 \mid (n^5 - n)$.

Solution: Recall that $5|(n^5 - n)$ if there exists a $l_n \in \mathbb{Z}$ so that $5l = n^5 - n$.

Intial Step: Does $5|(0-0)$? Yes $5|0$ since $5(0) = 0$.

Inductive Step: Assume that $5|(k^5 - k)$. Want to show that $5|((k+1)^5 - (k+1))$. Since $5|(k^5 - k)$ we know there is a $l_k \in \mathbb{Z}$ so that $5l_k = k^5 - k(\star)$. Let's Play!

$$\begin{aligned}
(k+1)^5 - (k+1) &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1) && \text{(Mad props to Pascal)} \\
&= (5k^4 + 10k^3 + 10k^2 + 5k) + (k^5 - k) \\
&= (5k^4 + 10k^3 + 10k^2 + 5k) + 5l_k \\
&= 5[k^4 + 2k^3 + 2k^2 + k + l_k]
\end{aligned}$$

Therefore $5|((k+1)^5 - (k+1))$ as we were able to find an integer $l_{k+1} = k^4 + 2k^3 + 2k^2 + k + l_k$ so that $5l_{k+1} = ((k+1)^5 - (k+1))$

Therefore by mathematical induction we know that for all $n \in \mathbb{Z}_{\geq 0}$, $5 \mid (n^5 - n)$.

Example 27. Prove by induction that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$.

Solution: Initial Step: Let $n = 1$

$$\frac{1}{6}(2)(3) = 1 = 1^2 = \sum_{i=1}^{1} i^2$$

Inductive Step: Suppose that $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k}{6}(k+1)(2k+1)(\star)$ now consider

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k}{6}(k+1)(2k+1) + (k+1)^2 && \text{(using } (\star)\text{)} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)(2k+3)(k+2)}{6} \\
&= \frac{k+1}{6}(k+2)(2k+3)
\end{aligned}$$

Therefore by induction $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$. for all $n \in \mathbb{N}$.

Example 28. Prove that if $n \in \mathbb{N}$, then $4^{2n} + 10n - 1$ is divisible by 25.

Solution: We will use induction. For $n \in \mathbb{N}$, let $A(n)$ be the statement $25 \mid (4^{2n} + 10n - 1)$.

Base case: Since $4^{2\cdot 1} + 10 \cdot 1 - 1 = 16 + 10 - 1 = 15$, we see that $4^{2n} + 10n - 1$ is divisible by 3 when $n = 1$ and so $A(1)$ is true.

Induction step: Suppose that $A(n)$ is true for some arbitrary $n \in \mathbb{N}$. Then, there is some $k \in \mathbb{Z}$ such that $4^{2n} + 10n - 1 = 25k$, or rearranging we have $4^{2n} = 25k + 1 - 10n$. Then we calculate that

$$\begin{aligned}
4^{2(n+1)} + 10(n+1) - 1 &= 4^{2n} \cdot 16 + 10n + 9 \\
&= 16(25k + 1 - 10n) + 10n + 9 && \text{(by the induction hypothesis)} \\
&= 25(16k) + 16 - 160n + 10n + 9 \\
&= 25(16k) - 150n + 25 \\
&= 25(16k - 6n + 1).
\end{aligned}$$

Since n and k are integers, so is $16k - 6n + 1$, so this means that $4^{2(n+1)} + 10(n+1) - 1$ is divisible by 25. This proves $A(n+1)$ is true when $A(n)$ is true, so by mathematical induction we have that $A(n)$ is true for every $n \in \mathbb{N}$.

Even Odd Proofs

Example 29 (Prove using Contradiction)**.** Suppose $a \in \mathbb{Z}$. Prove that if a^2 is even, then a is even.

Solution: Suppose that a^2 is even and yet a is odd. Since a is odd it can be expressed as $a = 2k + 1$ for some integer k. Then we can observe a^2.

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

And therefore a^2 is odd since it can be expressed as $2l + 1$ where $l = 2k^2 + 2k$. Since $k \in \mathbb{Z}$ we know $l \in \mathbb{Z}$. But a^2 cannot be both even and odd, thus we get a contradiction. So a must be even.

Example 30. Prove that If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Solution: Let's try a proof by contradiction. Assume that $a, b \in \mathbb{Z}$ and that $a^2 - 4b = 2$. Then we can rewrite it as $a^2 = 4b + 2 = 2(2b + 1)$ and so a^2 is even. By **Example 29** we know the a is even, that is there exists $k \in \mathbb{Z}$ so that $a = 2k$. Therefore:

$$\begin{aligned} a^2 &= 4b + 2 \\ (2k)^2 &= 4b + 2 \\ 4k^2 &= 4b + 2 \\ 2k^2 &= 2b + 1 \\ 2k^2 - 2b &= 1 \\ 2(k^2 - b) &= 1 \end{aligned}$$

And since $k, b \in \mathbb{Z}$ we know that $k^2 - b \in \mathbb{Z}$ and therefore 1 is even... which is a contradiction. $1 = 2(0) + 1$ which means it is odd and integers can not be both even and odd. Therefore if $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Example 31. Let n be an integer. Show that n^2 is odd if and only if n is odd.

Solution: The above statement is a biconditional statement, i.e., we need to prove two implications. First, we will prove that if n is an odd integer, then n^2 is odd. Indeed, assume $n \in \mathbb{Z}$ is odd. Then, there exists an integer k such that $n = 2k + 1$. Thus, $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where m is defined by $m = 2k^2 + 2k$. Since $\mathbb{Z}$ is closed under addition and multiplication, m is an integer, thus we can conclude that n^2 is indeed odd.

Now, we need to prove the converse, i.e. we need to show that if $n \in \mathbb{Z}$ and n^2 is odd, then n is odd. We will instead prove the equivalent contrapositive statement: "If n is an even integer, then n^2 is even." Indeed, assume $n \in \mathbb{Z}$ is even. Then, there exists an integer k such that $n = 2k$. Thus, $n^2 = 4k^2 = 2(2k^2) + 1 = 2m$, where m is defined by $m = 2k^2$ and m is an integer since $k \in \mathbb{Z}$. Thus, we can conclude that n^2 is indeed even.

We proved both directions, therefore we can conclude that an integer n is odd if and only if n^2 is odd.

Example 32. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Solution: Assume, by way of contradiction, that $a, b, c \in \mathbb{Z}$ satisfy $a^2 + b^2 = c^2$ and a and b are both odd. Then there exist $k, m \in \mathbb{Z}$ such that $a = 2k+1$ and $b = 2m+1$. Thus, $c^2 = 4k^2 + 4k + 1 + 4m^2 + 4m + 1 = 2(2k^2 + 2k + 2m^2 + 2m + 1) = 2s$, where $s = 2k^2 + 2k + 2m^2 + 2m + 1$ is an odd integer. Then, c^2 is even and therefore, c is even. Thus, $c = 2n$ for some integer n. Substitute this representation of c into $c^2 = 2s$ to arrive at $4n^2 = 2s$, which is equivalent to $2n^2 = s$, which implies that s is even. However, this contradicts the definition of s. Thus, the assumption that both a and b are odd leads to a contradiction, so at least one of them must be even.

Example 33. Prove that there is no largest even integer.

Solution: Suppose that there is an consider a proof by contradiction. Call it m and since it is even we know $m = 2k$ for some $k \in \mathbb{Z}$. Now consider $m + 2 = 2k + 2 + 2(k + 1)$. This shows that $m + 2$ is even and we know $m < m + 2$ since $0 < 2$ so therefore we get a contradiction and there is no largest even integer.

Example 34. Prove the following claim

Claim: Suppose $a \in \mathbb{Z}$. If $a^2 - 2a + 7$ is even, then a is odd.

Real Analysis

Indexed Sets

Example 35. Let $B_1 = \{1, 2\}, B_2 = \{2, 3\}, \dots, B_{10} = \{10, 11\}$; that is, $B_i = \{i, i + 1\}$ for some $i = 1, 2, \dots, 10$. Determine the following:

(i) $\displaystyle\bigcup_{i=1}^{5} B_i$

(ii) $\displaystyle\bigcup_{i=1}^{10} B_i$

(iii) $\displaystyle\bigcup_{i=3}^{7} B_i$

(iv) $\displaystyle\bigcup_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

(v) $\displaystyle\bigcap_{i=1}^{10} B_i$

(vi) $B_i \cap B_{i+1}$

(vii) $\displaystyle\bigcap_{i=j}^{j+1} B_i$, where $1 \leq j < 10$

(viii) $\displaystyle\bigcap_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

Solution:

(i) $\displaystyle\bigcup_{i=1}^{5} B_i = \{1,2,3,4,5,6\}$

(ii) $\displaystyle\bigcup_{i=1}^{10} B_i = \{1,2,3,\ldots,10,11\}$

(iii) $\displaystyle\bigcup_{i=3}^{7} B_i = \{3,4,5,6,7,8\}$

(iv) $\displaystyle\bigcup_{i=j}^{k} B_i = \{j, j+1,\ldots,k,k+1\}$, where $1 \le j \le k \le 10$.

(v) $\displaystyle\bigcap_{i=1}^{10} B_i = \emptyset$

(vi) $B_i \cap B_{i+1} = \{i+1\}$

(vii) $\displaystyle\bigcap_{i=j}^{j+1} B_i = \{j+1\}$, where $1 \le j < 10$

$(viii)$ Assume $1 \le j \le k \le 10$. Then

$$\bigcap_{i=j}^{k} B_i = \begin{cases} \{j, j+1\}, & \text{if } k = j \\ \{j+1\}, & \text{if } k = j+1 \\ \emptyset, & \text{if } k \ge j+2. \end{cases}$$

Example 36. Prove that $\displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2] = [3,5]$.

Solution: First, we will show that $[3,5] \subseteq \displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2]$. Indeed, let $y \in [3,5]$ be a generic element. Then, $3-(1/x)^2 \le 3 \le y \le 5 \le 5+(1/x)^2$ for all $x \in \mathbb{N}$. Thus, $y \in [3-(1/x)^2, 5+(1/x)^2]$ for all $x \in \mathbb{N}$, i.e. $y \in \displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2]$. This proves $[3,5] \subseteq \displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2]$.
Now, to prove $\displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2] \subseteq [3,5]$, we will use proof by contradiction. Assume there exists $y \in \displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2]$ such that $y \notin [3,5]$. If $y \notin [3,5]$, then $y < 3$ or $y > 5$. We will consider the case $y > 5$ (the case $y < 3$ is analogous). By the Archimedean property of $\mathbb{R}$, there exists $x_0 \in \mathbb{N}$ such that $x_0 > \frac{1}{y-5}$ and since $x_0^2 \ge x_0$, we have $y > 5 + \frac{1}{x_0^2}$. This implies $y \notin [3-(1/x_0)^2, 5+(1/x_0)^2]$, which contradicts the assumption that $y \in \displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2]$. Thus, we can conclude that $\displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2] \subseteq [3,5]$.
Finally, this implies $\displaystyle\bigcap_{x\in\mathbb{N}} [3-(1/x)^2, 5+(1/x)^2] = [3,5]$.

Bounded and Unbounded Sets

Example 37. Discuss whether the following sets are bounded or not bounded.

(a) $A = \{-2, -1, 1/2\}$.

(b) $B = (-\infty, \sqrt{2})$.

(c) $C = \{1/2, 3/2, 5/2, 7/2, 9/2, \dots\} = \{\frac{2n-1}{2} | n \in \mathbb{N}\}$.

(d) $D = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{N} \right\}$

(e) $E = \left\{ \frac{-1}{n} \ : \ n \in \mathbb{Q} \setminus \{0\} \right\}$

Solution:

(a) $A = \{-2, -1, 1/2\}$ is bounded. Indeed, take $M = 2$. Then $|x| \leq M$ for all $x \in A$.

(b) $B = (-\infty, \sqrt{2})$ is unbounded. Note that $-n \in B$ for all $n \in \mathbb{N}$. If we assume, by way of contradiction, B is bounded, then there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in B$. However, by the Archimedean property of real numbers, there exists $n \in \mathbb{N}$ such that $n > M$. Since, $-n \in B$ and $|-n| \not\leq M$, we have reached a contradiction, thus B is indeed unbounded.

(c) $C = \{1/2, 3/2, 5/2, 7/2, 9/2, \dots\} = \{\frac{2n-1}{2} | n \in \mathbb{N}\}$ is unbounded. Assume, by way of contradiction, that C is bounded. Then there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in C$. However, by the Archimedean property of real numbers, there exists $n \in \mathbb{N}$ such that $n > M + 1$. Since, $\frac{2n-1}{2} \in C$ and $\frac{2n-1}{2} > \frac{2(M+1)-1}{2} > M$, we have reached a contradiction, thus C is indeed unbounded.

(d) $D = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{N} \right\}$ is bounded. Note that $\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, since $n \geq 1$ for all natural numbers.

(e) $E = \left\{ \frac{-1}{n} \ : \ n \in \mathbb{Q} \setminus \{0\} \right\}$ is unbounded. Assume, to the contrary, that E is bounded. Then, there exists $M \in \mathbb{R}$ such that $|x| < M$ for all $x \in E$. On the other hand, by the Archimedean property of natural numbers, there exists $n \in \mathbb{N}$ such that $n > M$. Take $q = \frac{1}{n}$ and note that $q \in \mathbb{Q}$, so $\frac{-1}{q} \in E$. Thus, $\left| \frac{-1}{q} \right| < M$ (by the assumption that M provides a bound for the elements in E). On the other hand, $\left| \frac{-1}{q} \right| = \frac{1}{q} = n > M$, a contradiction. We can conclude that the set E is not bounded.

Sequences

Example 38. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{ \frac{3n+1}{7n-4} \right\}_{n \in \mathbb{N}}$

(b) $\left\{ \sin(\frac{n\pi}{4}) \right\}_{n \in \mathbb{N}}$

(c) $\{(1 + 1/n)^2\}_{n \in \mathbb{N}}$

(d) $\{(-1)^n n\}_{n \in \mathbb{N}}$

(e) $\left\{\sqrt{n^2+1}-n\right\}_{n\in\mathbb{N}}$

Solution:

(a) $\left\{\frac{3n+1}{7n-4}\right\}_{n\in\mathbb{N}}$ converges to $\frac{3}{7}$. We could justify it by rewriting the terms of the sequence in the form $\frac{3+\frac{1}{n}}{7-\frac{4}{n}}$ and noting that $\frac{1}{n}$ goes to 0 as n goes to infinity.

(b) $\left\{\sin(\frac{n\pi}{4})\right\}_{n\in\mathbb{N}}$ diverges. Note that

$$\left\{\sin(\frac{n\pi}{4})\right\}_{n\in\mathbb{N}} = \{\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, ...\}.$$

(c) $\{(1+1/n)^2\}_{n\in\mathbb{N}}$ converges to 1, since $\frac{1}{n}$ converges to 0 as n goes to infinity.

(d) $\{(-1)^n n\}_{n\in\mathbb{N}}$ diverges because the odd terms go to $-\infty$ and the even terms go to ∞.

(e) $\left\{\sqrt{n^2+1}-n\right\}_{n\in\mathbb{N}}$ converges to 0. Indeed,

$$0 \leq \sqrt{n^2+1}-n = (\sqrt{n^2+1}-n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n}.$$

Note that $\frac{1}{2n}$ goes to 0 as n goes to infinity, so by the Squeeze Theorem, the original sequence converges to 0.

Example 39. Using the definition of convergence, that is, an $\varepsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0.$

(b) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0.$

(c) $\lim\limits_{n\to\infty} \dfrac{n}{2n+1} = \dfrac{1}{2}.$

Solution:

(a) To prove $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, let $\varepsilon > 0$ be given, arbitrary. Define $N = \lceil\frac{1}{\varepsilon}\rceil$ (this is the smallest integer larger than $\frac{1}{\varepsilon}$). Assume $n \in \mathbb{N}$ is such that $n > N$. Then,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\left|\frac{1}{n} - 0\right| \leq \varepsilon$. We can conclude that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0.$

(b) To prove $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, let $\varepsilon > 0$ be given, arbitrary. Define $N = \lceil \frac{1}{\varepsilon^2} \rceil$. Assume $n \in \mathbb{N}$ is such that $n > N$. Then,

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\left| \frac{1}{\sqrt{n}} - 0 \right| \leq \varepsilon$. We can conclude that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

(c) To prove $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$, let $\varepsilon > 0$ be given, arbitrary. Define $N = \lceil \frac{1}{\varepsilon} \rceil$. Assume $n \in \mathbb{N}$ is such that $n > N$. Then,

$$\left| \frac{n}{2n+1} - \frac{1}{2} \right| = \frac{1}{2n+1} < \frac{1}{2N+1} < \frac{1}{N} \leq \varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\left| \frac{n}{2n+1} - \frac{1}{2} \right| \leq \varepsilon$. We can conclude that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

Example 40. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{3 + \frac{(-1)^n 2}{n}\right\}_{n\in\mathbb{N}}$

(b) $\left\{\frac{n^2-2n+1}{n-1}\right\}_{n\in\mathbb{N}\setminus\{1\}}$

(c) $\left\{\frac{n}{n+1}\right\}_{n\in\mathbb{N}}$

Solution:

(a) $\left\{3 + \frac{(-1)^n 2}{n}\right\}_{n\in\mathbb{N}}$ converges to 3. The reason is that $\frac{(-1)^n 2}{n}$ goes to 0 as n approaches ∞.

(b) $\left\{\frac{n^2-2n+1}{n-1}\right\}_{n\in\mathbb{N}\setminus\{1\}}$ diverges (in particular, it approaches ∞ as n goes to ∞.) Note that

$$\frac{n^2-2n+1}{n-1} = \frac{(n-1)^2}{n-1} = n - 1 \text{ whenever } n \neq 1.$$

(c) $\left\{\frac{n}{n+1}\right\}_{n\in\mathbb{N}}$ converges to 1. We can rewrite the terms of the sequence in the following way:

$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Now, note that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Thus, $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Example 41. Using the definition of convergence, that is, an $\varepsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\lim_{n\to\infty}\left(3+\frac{2}{n^2}\right)=3.$

(b) $\lim_{n\to\infty}\frac{\sin(n)}{2n+1}=0.$

Solution:

(a) To prove $\lim_{n\to\infty}\left(3+\frac{2}{n^2}\right)=3$, let $\varepsilon>0$ be given, arbitrary. Define $N=\lceil\sqrt{\frac{2}{\varepsilon}}\rceil$ and let $n\in\mathbb{N}$ be such that $n>N$. Then

$$\begin{aligned}\left|3+\frac{2}{n^2}-3\right| &= \frac{2}{n^2}<\frac{2}{N^2} && \text{since } n>N\\ &\le \frac{(\sqrt{\varepsilon/2})^2}{2} && \text{since } N=\left\lceil\sqrt{\frac{2}{\varepsilon}}\right\rceil \ge \sqrt{\frac{\varepsilon}{2}}\\ &= \varepsilon.\end{aligned}$$

Thus, for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that whenever $n\in\mathbb{N}$ satisfies $n>N$, we have $\left|3+\frac{2}{n^2}-3\right|<\varepsilon$, i.e., $\lim_{n\to\infty}\left(3+\frac{2}{n^2}\right)=3.$

(b) To prove $\lim_{n\to\infty}\frac{\sin(n)}{2n+1}=0$, let $\varepsilon>0$ be given, arbitrary. Define $N=\lceil\frac{1}{\varepsilon}\rceil$ and let $n\in\mathbb{N}$ be such that $n>N$. Then

$$\begin{aligned}\left|\frac{\sin(n)}{2n+1}-0\right| &= \frac{|\sin(n)|}{2n+1}\le\frac{1}{2n+1} && \text{since } |\sin(n)|\le 1\\ &< \frac{1}{2N+1} && \text{since } n>N\\ &\le \frac{1}{N}\le\frac{1}{\frac{1}{\varepsilon}} && \text{since } N\ge\frac{1}{\varepsilon}=\varepsilon.\end{aligned}$$

Thus, for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that whenever $n\in\mathbb{N}$ satisfies $n>N$, we have $\left|\frac{\sin(n)}{2n+1}-0\right|<\varepsilon$, i.e., $\lim_{n\to\infty}\frac{\sin(n)}{2n+1}=0.$

Example 42. Prove that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ does not converge.

Solution: Suppose that is does (working towards a contradiction). That is: $\exists L\in\mathbb{R}$, so that $\forall\varepsilon>0,\exists N\in\mathbb{N}$ such that for all $n\ge N$ we have $|(-1)^n-L|<\varepsilon$. Since this must be true for all $\varepsilon>0$ it must certainly be true for $\varepsilon=.5$. Now I will make my own mini theorem that we will use later:

Theorem ($\star$): For all $M \in \mathbb{N}$ there exists an odd number greater than M and there exists an even number greater than M.

This will give us that no matter what $N \in \mathbb{N}$ is chosen there will be some $n \geq N$ that is even and another that is odd. That means that both of the following inequalities must be true:

$$|1 - L| < .5 \qquad\qquad |-1 - L| < .5$$

So our contradiction lies with the question: What is L? Expanding these absolute value inequalities we get:

$$\begin{aligned} |1 - L| &< .5 \\ |L - 1| &< .5 \\ -.5 < L - 1 &< .5 \\ .5 < L &< 1.5 \end{aligned} \qquad\qquad \begin{aligned} |-1 - L| &< .5 \\ |L + 1| &< .5 \\ -.5 < L + 1 &< .5 \\ -1.5 < L &< -.5 \end{aligned}$$

So by the first compound inequality we have $L \in (.5, 1.5)$ and by the second $L \in (-1.5, -.5)$. But these sets have an empty intersection so it can not be in both. Giving us a contradiction, Therefore no limit L exists and so the sequence cannot converge.

Example 43. Use the formal definition of the limit of a sequence to prove that

$$\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}.$$

Solution:

Scratch Work:

$$\begin{aligned} \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| &< \varepsilon \\ \left|\frac{3(2n-1) - 2(3n+2)}{3(3n+2)}\right| &< \varepsilon \\ \left|\frac{6n-3-6n-4}{9n+6}\right| &< \varepsilon \\ \left|\frac{-7}{9n+6}\right| &< \varepsilon \\ \frac{7}{9n+6} &< \varepsilon \\ \frac{7}{9n+6} < \frac{7}{9n} &< \varepsilon \end{aligned}$$

Take $N > \dfrac{7}{9\varepsilon}$ (possible by the A.P.). Then if $n \geq N$ we have $n > \dfrac{7}{9\varepsilon}$. This implies that

$$\begin{aligned}
\frac{7}{9\varepsilon} &< n \\
\frac{7}{9n} &< \varepsilon && \text{both } n \text{ and } \varepsilon \text{ are greater than } 0 \\
\frac{7}{9n+6} &< \varepsilon && \text{since } 9n < 9n+6 \\
\left|\frac{-7}{9n+6}\right| &< \varepsilon \\
\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| &< \varepsilon
\end{aligned}$$

Giving us that $\lim\limits_{n\to\infty} \dfrac{2n-1}{3n+2} = \dfrac{2}{3}$ by definition as desired.

Open and Closed Sets

Example 44. Which of the following sets are open?

$$(i)(-3,3) \qquad (ii)(-4,5] \qquad (iii)(0,\infty) \qquad (iv)\{(x,y)\in\mathbb{R}^2 : y>0\}$$

$$(iv)\bigcup_{n=1}^{5}\left(-1+\frac{1}{n}, 1-\frac{1}{n}\right), \text{ where } n\in\mathbb{N} \qquad (v)\{x\in\mathbb{R} : |x-1|<2\}.$$

Solution:

$$(i)(-3,3) \qquad (ii)(-4,5] \qquad (iii)(0,\infty) \qquad (iv)\{(x,y)\in\mathbb{R}^2 : y>0\}$$

$$(iv)\bigcup_{n=1}^{5}\left(-1+\frac{1}{n}, 1-\frac{1}{n}\right), \text{ where } n\in\mathbb{N} \qquad (v)\{x\in\mathbb{R} : |x-1|<2\}.$$

Example 45. Let $A = [2,5]$. Discuss whether A^c an open set.

Solution: $A^c = (-\infty,2)\cup(5,\infty)$ is an open set. Indeed, if $x\in A^c$, then $x\in(-\infty,2)$ or $x\in(5,\infty)$. If $x\in(-\infty,2)$, take $r = 2-x$. Then we claim that $(x-r,x+r)\subseteq(-\infty,2)$. Indeed, $x<2$, so $r>0$ and $x+r = x+2-x = 2$. Thus, if $y\in(x-r,x+r)$, then $y<x+r=2$, thus, $y\in(-\infty,2)$, i.e. $(x-r,x+r)\subseteq(-\infty,2)$. Similarly, if $x\in(5,\infty)$, take $r = x-5$ and show that $(x-r,x+r)\subseteq(5,\infty)$. In either case, for every $x\in A^c$, there exists $r>0$, such that $(x-r,x+r)\subseteq A^c$, thus A^c is an open set.

Example 46. Discuss which of the following sets are open.

(a) $(2,3)$.

(b) $(-4,8]$.

(c) $[1,3)$.

(d) $(-\infty,\infty)$.

(e) $[1,5]\cap[2,3]$.

Solution:

(a) $(2,3)$ is an open set.

(b) $(-4,8]$ is not open. For the point $8\in(-4,8]$, there is no positive value r such that $(8-r,8+r)\subseteq(-4,8]$.

(c) $[1,3)$ is not open. For the point $1\in[1,3)$, there is no positive value r such that $(1-r,1+r)\subseteq[1,3)$.

(d) $(-\infty,\infty)$ is open.

(e) $[1,5]\cap[2,3]=[2,3]$ is not open. For the point $2\in[2,3]$, for example, there is no positive value r such that $(2-r,2+r)\subseteq[2,3]$.

Example 47. Which of the following sets are closed? Justify your answer.

$$(i)\ A=[2,5]\qquad (ii)\ B=(-1,0)\cup(0,1)\qquad (iii)\ C=\{x\in\mathbb{R}:|x-1|<2\}$$
$$(iv)\ D=\{-2,-1,0,1,2\}\qquad (v)\ \mathbb{Z}$$

Solution:

1. $A=[2,5]$ is closed since $A^c=(-\infty,2)\cup(5,\infty)$ is open. See Example 45 for the proof.
2. $B=(-1,0)\cup(0,1)$ is not closed, since $B^c=(-\infty,-1]\cup\{0\}\cup[1,\infty)$ is not open. Indeed, consider the point $0\in B^c$, for example. There is no $r>0$ such that $(-r,r)\subseteq B^c$.
3. $C=\{x\in\mathbb{R}:|x-1|<2\}=\{x\in\mathbb{R}:-2<x-1<2\}=\{x\in\mathbb{R}:-1<x<3\}=(1,3)$. Thus, $C^c=(-\infty,-1]\cup[3,\infty)$. Thus, C is not closed, as C^c is not open. To show C^c is not open, consider, for example, $-1\in C^c$. There is no $r>0$ such that $(-1-r,-1+r)\subseteq C^c$.
4. $D=\{-2,-1,0,1,2\}$ is a closed set. Indeed, consider $D^c=(-\infty,-2)\cup(-2,-1)\cup(-1,0)\cup(0,1)\cup(1,2)\cup(2,\infty)$. We will prove below that the union of open sets is open and that any set of the form $(a,b)\subseteq\mathbb{R}$, where $a,b\in\mathbb{R}$ with $a<b$ or $a=-\infty$ or $b=\infty$ is open. This proves D^c is open, so D is closed.
5. $\mathbb{Z}$ is closed, since $\mathbb{Z}^c=\displaystyle\bigcup_{i=-\infty}^{\infty}(i,i+1)$ is an open set, being the union of open sets.

Lemma 1. Any interval $(a, b) \subseteq \mathbb{R}$ with with $a, b \in \mathbb{R}$ such that $a < b$, or $a = -\infty$, or $b = \infty$ is an open set in $\mathbb{R}$.

Proof. Let us first consider the case when $a, b \in \mathbb{R}$ with $a < b$. Assume $x \in (a, b)$ is a generic element. Define $r = \min\{x-a, b-x\}$. Thus, $r > 0$. We claim $(x-r, x+r) \subseteq (a, b)$. Indeed, take any $y \in (x-r, x+r)$. Then

$$\begin{aligned} a &\le x - r && \text{since } r = \min\{x-a, b-x\},\ \text{so } r \le x - a,\ \text{equivalently, } a \le x - r \\ &\le y \le x + r && \text{since } y \in (x-r, x+r) \\ &\le b && \text{since } r = \min\{x-a, b-x\},\ \text{so } r \le b - x,\ \text{equivalently, } x + r \le b. \end{aligned}$$

Thus, for any $x \in (a, b)$, there exists $r > 0$ such that $(x - r, x + r) \subseteq (a, b)$, i.e., (a, b) is an open set in $\mathbb{R}$. Modify the proof, to show that for any $a, b \in \mathbb{R}$, $(-\infty, b)$, (a, ∞) and $(-\infty, \infty)$ are all open sets in $\mathbb{R}$.

Lemma 2. Let $\{A_i\}_{i\in I}$ be a collection of open sets with $A_i \subseteq \mathbb{R}$ for all $i \in I$, where I is an indexing set. Then $\displaystyle\bigcup_{i\in I} A_i$ is an open set.

Proof. Assume $x \in \displaystyle\bigcup_{i\in I} A_i$. Then $x \in A_k$ for some $k \in I$. Since A_k is an open set, there exists $r > 0$ such that $(x-r, x+r) \subseteq A_k \subseteq \displaystyle\bigcup_{i\in I} A_i$. Thus, for every $x \in \displaystyle\bigcup_{i\in I} A_i$, there exists $r > 0$ such that $(x-r, x+r) \subseteq \displaystyle\bigcup_{i\in I} A_i$, i.e., $\bigcup_{i\in I} A_i$ is an open set.

Linear Algebra

Vector Spaces

Example 48. Show that the set $\mathbb{R}^2$ over $\mathbb{R}$ is not a vector space under the following definitions for vector addition and scalar multiplication:

$$x + y := (x_1 - y_1, x_2 - y_2)$$

and

$$\lambda x := (\lambda x_1, \lambda x_2),$$

where $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$.

Solution: Under these definitions, commutativity of addition fails: consider $(1, 1)$ and $(1, 2) \in \mathbb{R}^2$, then for the above definition of addition,

$$(1, 1) + (1, 2) = (0, -1)$$

but

$$(1, 2) + (1, 1) = (0, 1)$$

which are not equal, so commutativity of addition fails.
Alternatively, distributivity of scalar multiplication also fails: for any $(x_1, x_2) \neq (0, 0)$,

$$(1 + (-1))(x_1, x_2) = 0(x_1, x_2) = (0, 0)$$

while

$$1(x_1, x_2) + (-1)(x_1, x_2) = (x_1, x_2) + (-x_1, -x_2) = (2x_1, 2x_2)$$

which are not equal.

Example 49. Under the usual matrix operations, is the set

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

a vector space over $\mathbb{R}$? Justify your answer.

Solution: The above set is a vector space over $\mathbb{R}$, let's call the set X. By Proposition 8.8 (in the supplementary document), we know the set V which is the space of all 2×2 matrices with real entries is a vector space, under the same definition of addition and scalar multiplication as X. Then, by Proposition 8.5, if we can show that X is closed under addition and scalar multiplication, this will prove X is a subspace of V. Finally, we see that a subspace of a vector space is also a vector space by Definition 8.4, so this will prove that X is a vector space itself.
Closed under addition: Let A_1, $A_2 \in X$, then for some real numbers a_1, a_2, b_1, b_2, c_1, and c_2, we have

$$A_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}.$$

Then, since the sum of two real numbers is again a real number,

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & 0 \\ b_1 + b_2 & c_1 + c_2 \end{pmatrix} \in X$$

by definition, so X is closed under addition.
Closed under scalar multiplication: Let $A \in X$ and $\lambda \in \mathbb{R}$, then

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

for some a, b, $c \in \mathbb{R}$. Then, since the product of two real numbers is again a real number,

$$\lambda A \begin{pmatrix} \lambda a & 0 \\ \lambda b & \lambda c \end{pmatrix} \in X$$

so X is closed under scalar multiplication. This completes our proof.

Example 50. Define the set $V = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_2 \geq 2x_1 + 1\}$. Sketch a picture of the set V inside of the plane, $\mathbb{R}^2$. Is V a vector space over $\mathbb{R}$? Justify your answer!

Solution: The set V with vector addition and scalar multiplication, defined in the usual way, is not a vector space, since Axiom (v) in Definition 8.1 is violated, i.e., the set V does not contain the zero element of addition. Indeed, $(0, 0)$ does not satisfy $0 \geq 2 \cdot 0 + 1$, i.e. $(0, 0) \notin V$.

Example 51. Define

$$V = \{p \in \mathbb{P}_2 \;:\; \forall x \in \mathbb{R},\ p'(1) = 0\}.$$

Is V a vector space over $\mathbb{R}$?

Solution: The set V is a vector space over $\mathbb{R}$. If we knew that $\mathbb{P}_2$ was a vector space over $\mathbb{R}$, then we would only need to check that V is closed under addition and scalar multiplication (Proposition 8.5). We do not yet know this, however, so we will need to verify all the axioms in Definition 8.1.

(i) Closure of addition: Let $p, q \in V$. Then, $p + q \in \mathbb{P}_3$ and $(p + q)'(1) = p'(1) + q'(1) = 0 + 0 = 0$, so $p + q \in V$.

(ii) Closure of scalar multiplication: Let $p \in V$ and $c \in \mathbb{R}$. Then $cp \in \mathbb{P}_3$ and $(cp)'(1) = cp'(1) = c \cdot 0 = 0$, so $cp \in V$.

The following properties are satisfied for all polynomials in $\mathbb{P}_3$.

(iii) Associativity of addition: Let $p, q, r \in V$, then $p + (q + r) = (p + q) + r$, by definition of function addition and associativity of $\mathbb{R}$.

(iv) Commutativity of addition: Let $p, q \in V$, then $p + q = q + p$.

(v) Identity element of addition: There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v$ for all $v \in V$ - indeed, the zero element of V is given by the constant zero polynomial $o(x) = 0$ for all $x \in R$. Note that $o \in V$.

(vi) Inverse elements of addition: For every $p \in V$, there exists an element $-p \in V$, called the additive inverse of p, such that $p + (-p) = 0$. This is indeed, the case - given $p \in V$, define $-p$ by $(-p)(x) = -(p(x))$ for all $x \in \mathbb{R}$. Note that if $p \in V$, then $-p \in V$.

(vii) Compatibility of scalar multiplication with scalar multiplication: Let $a, b \in \mathbb{R}$ and $p \in V$, then by definition of multiplication of polynomial by a scalar and associativity of reals, $a(bp) = (ab)p$.

(viii) Identity element of scalar multiplication: $1 \in \mathbb{R}$ satisfies $1 \cdot p = p$ for all $p \in V$.

(ix) Distributivity of scalar multiplication with respect to vector addition: Let $a \in \mathbb{R}$ and $p, q \in V$. Then by the definition of multiplication of a polynomial by a scalar and by the distributivity of reals, we have $a(p + q) = ap + aq$.

(x) Distributivity of scalar multiplication with respect to field addition: Similarly, for $a, b \in \mathbb{R}$ and $p \in V$, $(a + b)p = ap + bp$.

Since all the axioms for a vector space a satisfied, we can conclude that V is a vector space.

Example 52. Find the additive inverse, in the vector space, of the following:

(a) In $\mathbb{P}_3$, of the element $-3 - 2x + x^2$.

Solution: $3 + 2x - x^2$

(b) In the space of 2×2 matrices, of the element $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$.

Solution: $\begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$

(c) In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x, the element $3e^x - 2e^{-x}$.

Solution: $-3e^x + 2e^{-x}$

You may assume that these vector spaces are defined over $\mathbb{R}$ and that in each case, natural definitions of addition and scalar multiplication hold.

Example 53. Show that the set of linear polynomials $\mathbb{P}_1 = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations is a vector space over $\mathbb{R}$.

Solution:

(i) Closure of addition: $v + w \in V$, that is "+" is a function such that $+ : V \times V \to V$.

Take $p(x) = c_0 + c_1 x \in \mathbb{P}_1$ and $q(x) = d_0 + d_1 x \in \mathbb{P}_1$ Then $(p + q)(x) = (c_0 + d_0) + (c_1 + d_1)x \in \mathbb{P}_1$ since the reals a closed under addition.

(ii) Closure of scalar multiplication: $av \in V$, that is "multiplication" is a function with "multiplication" $: \mathbb{R} \times V \to V$.

Take $p(x) = c_0 + c_1 x \in \mathbb{P}_1$ and $k \in \mathbb{R}$ then $(kp)(x) = (kc_0) + (kc_1)x \in \mathbb{P}_1$ since the reals are closed under multiplication.

(iii) Associativity of addition: $u + (v + w) = (u + v) + w$.

Take $p(x) = c_0 + c_1 x \in \mathbb{P}_1$ and $q(x) = d_0 + d_1 x \in \mathbb{P}_1$ and $r(x) = b_0 + b_1 x \in \mathbb{P}_1$ then

$$\begin{aligned} p(x) + (q(x) + r(x)) &= (c_0 + c_1 x) + ((d_0 + d_1 x) + (b_0 + b_1 x)) \\ &= (c_0 + c_1 x) + ((d_0 + b_0) + (d_1 + b_1)x) \\ &= (c_0 + (d_0 + b_0)) + (c_1 + (d_1 + b_1))x \\ &= ((c_0 + d_0) + b_0) + ((c_1 + d_1) + b_1)x \\ &= ((c_0 + d_0) + (c_1 + d_1)x) + (b_0 + b_1 x) \\ &= (p(x) + q(x)) + r(x) \end{aligned}$$

(iv) Commutativity of addition: $u + v = v + u$.

Take $p(x) = c_0 + c_1x \in \mathbb{P}_1$ and $q(x) = d_0 + d_1x \in \mathbb{P}_1$ then

$$\begin{aligned} p(x) + q(x) &= (c_0 + c_1x) + (d_0 + d_1x) \\ &= (c_0 + d_0) + (c_1 + d_1)x \\ &= (d_0 + c_0) + (d_1 + c_1)x \\ &= (d_0 + d_1x) + (c_0 + c_1x) \\ &= q(x) + p(x) \end{aligned}$$

(v) Identity element of addition: There exists an element $0 \in V$, called the zero vector, such that $v+0 = v$ for all $v \in V$.

Take $p(x) = c_0 + c_1x \in \mathbb{P}_1$ and $o(x) = 0 + 0x \in \mathbb{P}_1$ then

$$\begin{aligned} p(x) + o(x) &= (c_0 + c_1x) + (0 + 0x) \\ &= (c_0 + 0) + (c_1 + 0)x \\ &= c_0 + c_1x \\ &= p(x) \end{aligned}$$

(vi) Inverse elements of addition: For every $v \in V$, there exists an element $-v \in V$, called the additive inverse of v, such that $v + (-v) = 0$.

Take $p(x) = c_0 + c_1x \in \mathbb{P}_1$ and define $n_p(x) = (-c_0) + (-c_1)x \in \mathbb{P}_1$ then

$$\begin{aligned} p(x) + n_p(x) &= (c_0 + c_1x) + (-c_0 + (-c_1)x) \\ &= (c_0 + (-c_0)) + (c_1 + (-c_1))x \\ &= 0 + 0x \end{aligned}$$

(vii) Compatibility of scalar multiplication with scalar multiplication: $a(bv) = (ab)v$.

Take $p(x) = c_0 + c_1x \in \mathbb{P}_1$ and $k, l \in \mathbb{R}$ then

$$\begin{aligned} k((lp)(x)) &= k(lc_0 + lc_1x) \\ &= (kl)c_0 + (kl)c_1x \\ &= (kl)(p(x)) \end{aligned}$$

(viii) Identity element of scalar multiplication: $1v = v$, where $1 \in \mathbb{R}$ denotes the multiplicative identity.

Take $p(x) = c_0 + c_1x \in \mathbb{P}_1$ and $1 \in \mathbb{R}$ then

$$\begin{aligned} 1(p(x)) &= (1c_0 + 1c_1x) \\ &= c_0 + c_1x \\ &= p(x) \end{aligned}$$

(ix) Distributivity of scalar multiplication with respect to vector addition: $a(u + v) = au + av$.

Take $p(x) = c_0 + c_1 x \in \mathbb{P}_1$ and $q(x) = d_0 + d_1 x \in \mathbb{P}_1$ and $k \in \mathbb{R}$ then

$$\begin{aligned} k(p(x) + q(x)) &= k((c_0 + c_1 x) + (d_0 + d_1 x)) \\ &= k((c_0 + d_0) + (c_1 + d_1)x) \\ &= k(c_0 + d_0) + k(c_1 + d_1)x \\ &= (kc_0 + kd_0) + (kc_1 + kd_1)x \\ &= (kc_0 + kc_1 x) + (kd_0 + kd_1 x) \\ &= k(c_0 + c_1 x) + k(d_0 + d_1 x) \\ &= kp(x) + kq(x) \end{aligned}$$

(x) Distributivity of scalar multiplication with respect to field addition: $(a + b)v = av + bv$.

Take $p(x) = c_0 + c_1 x \in \mathbb{P}_1$ and $k, l \in \mathbb{R}$ then

$$\begin{aligned} (k + l)p(x) &= (k + l)(c_0 + c_1 x) \\ &= (k + l)c_0 + (k + l)c_1 x \\ &= kc_0 + lc_0 + kc_1 x + lc_1 x \\ &= (kc_0 + kc_1 x) + (lc_0 + lc_1 x) \\ &= k(c_0 + c_1 x) + l(c_0 + c_1 x) \\ &= kp(x) + lp(x) \end{aligned}$$

Linear Maps

Example 54. Define the function $A : \mathbb{P}^3 \to \mathbb{P}^5$ by

$$A(p)(x) = x^2 p(x) \qquad \text{for } x \in \mathbb{R}.$$

Is A a linear function? Justify your answer.

Solution: A is a linear function. Let p, $q \in \mathbb{P}_3$, then for any $x \in \mathbb{R}$ we have

$$\begin{aligned} A(p + q)(x) &= x^2(p + q)(x) = x^2(p(x) + q(x)) \\ &= x^2 p(x) + x^2 q(x) = A(p)(x) + A(q)(x) \end{aligned}$$

so A is additive.
Also, suppose $a \in \mathbb{R}$, then for any $x \in \mathbb{R}$,

$$\begin{aligned} A(ap)(x) &= x^2(ap)(x) \\ &= ax^2 p(x) = aA(p)(x), \end{aligned}$$

so A satisfies the property of scalar multiplication. Therefore A is linear.

Example 55. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Solution:
(If T is linear, then $b = c = 0$): Suppose that T is linear. Then by the scalar multiplicity property, we must have for any x, y, $z \in \mathbb{R}$,

$$\begin{aligned} &0T(x, y, z) = T(0 \cdot (x, y, z)) = T(0, 0, 0) \\ \implies &(0, 0) = (2 \cdot 0 - 4 \cdot 0 + 3 \cdot 0 + b, 6 \cdot 0 + c \cdot 0 \cdot 0 \cdot 0) = (b, 0), \end{aligned}$$

therefore $b = 0$. Using this fact, we see that

$$\begin{aligned} T(1, 1, 1) &= (2 - 4 + 3, 6 + c) = (1, 6 + c) \\ T(2, 2, 2) &= (4 - 8 + 6, 12 + 8c) = (2, 12 + 8c). \end{aligned}$$

Thus again using the scalar multiplicity property, we must have

$$\begin{aligned} 2T(1, 1, 1) &= T(2(1, 1, 1)) = T(2, 2, 2) \\ \implies 2(1, 6 + c) = (2, 12 + c) &= (2, 12 + 8c) \\ \implies 12 + c &= 12 + 8c \\ \implies c &= 8c \\ \implies c &= 0. \end{aligned}$$

Therefore, we must have $b = c = 0$ if T is linear.
(If $b = c = 0$, then T is linear): Suppose $b = c = 0$. Then for any (x, y, z), $(u, v, w) \in \mathbb{R}^3$, we have

$$\begin{aligned} T((x, y, z) + (u, v, w)) &= T(x + u, y + v, z + w) \\ &= (2(x + u) - 4(y + v) + 3(z + w), 6(x + u)) \\ &= (2x - 4y + 3z + 2u - 4v + 3w, 6x + 6u) \\ &= (2x - 4y + 3z, 6x) + (2u - 4v + 3w, 6u) \\ &= T(x, y, z) + T(u, v, w), \end{aligned}$$

thus T is additive.
Also for any $a \in \mathbb{R}$,

$$\begin{aligned} T(a(x, y, z)) &= T(ax, ay, az) \\ &= (2ax - 4ay + 3az, 6ax) \\ &= a(2x - 4y + 3z, 6x) = aT(x, y, z), \end{aligned}$$

so T satisfies the property of scalar multiplication, hence is linear.

Example 56. For each of the following L, answer "yes" or "no", and briefly justify your answer:

(a) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = \sin(x)$, a linear function?

(b) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = |x|^{1/2}$, a linear function?

(c) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = 51.5x$, a linear function?

(d) Is $L : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $L(p) = 3p$ a linear function? Find the images under L of $p, q \in \mathbb{P}_3$ defined by $p(x) = x^3 - 7$ and $q(x) = 2x^2 + 3x + 5$.

Solution:

(a) $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = \sin(x)$ is not a linear function. For example, $\sin(\pi) = 0$, on the other hand, $\sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) = 2$. Thus, $\sin(\frac{\pi}{2} + \frac{\pi}{2}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2})$, thus, additivity is violated.

(b) $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = |x|^{1/2}$ is not a linear function. Consider $L(2) = \sqrt{2}$, on the other hand, $L(1) = 1$. Thus, $L(1 + 1) \neq L(1) + L(1)$. Thus, as above, L does not satisfy additivity and is thus not a linear function.

(c) $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = 51.5x$, is a linear function. Indeed, it satisfies

(i) Additivity: $L(x + y) = 51.5(x + y) = 51.5x + 51.5y = L(x) + L(y)$ for all $x, y \in \mathbb{R}$.

(ii) Scalar multiplication: for $x, c \in \mathbb{R}$, $L(cx) = 51.5(cx) = c51.5x = cL(x) + L(y)$.

Thus, L is a linear map.

(d) $L : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $L(p) = 3p$ is a linear function.

(i) Additivity: Let $p, q \in \mathbb{P}_3$ $L(p + q) = 3(p + q) = 3p + 3q = L(p) + L(q)$.

(ii) Scalar multiplication: Let $c \in \mathbb{R}$ and $p \in \mathbb{P}_3$ be arbitrary. $L(c) = 3(cp) = c(3p) = cL(p)$.

Thus, L is a linear map.

For, defined by $p(x) = x^3 - 7$, $L(p)(x) = 3x^3 - 21$ and for $q(x) = 2x^2 + 3x + 5$, $L(q)(x) = 6x^2 + 9x + 15$.

Example 57. Define the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the transformation

$$T\left((x_0, x_1)\right) = (x_0, 0),$$

where $(x_0, x_1) \in \mathbb{R}^2$. Is T a linear function? Justify your answer.

Solution: Take $(x_0, x_1) \in \mathbb{R}^2$ and $k \in \mathbb{R}$ then

$$\begin{aligned} T(k(x_0, x_1)) &= T((kx_0, kx_1)) \\ &= (kx_0, 0) \\ &= k(x_0, 0) \\ &= kT((x_0, x_1)) \end{aligned}$$

Take $(x_0, x_1) \in \mathbb{R}^2$ and $(y_0, y_1) \in \mathbb{R}^2$ then

$$\begin{aligned} T((x_0, x_1) + (y_0, y_1)) &= T((x_0 + y_0, x_1 + y_1)) \\ &= (x_0 + y_0, 0) \\ &= (x_0, 0) + (y_0, 0) \\ &= T((x_0, x_1)) + T((y_0, y_1)) \end{aligned}$$

So therefore T is linear by definition.

Abstract Algebra

Divisibility and Remainders

Example 58. Use the Euclidean Algorithm to find the greatest common divisor for each of the following pairs of integers:

(a) 51 and 288 $\quad$ (b) 357 and 629 $\quad$ (c) 180 and 252.

Solution: (a):

$$\begin{aligned} 288 &= 5 \cdot 51 + 33 \\ 51 &= 1 \cdot 33 + 18 \\ 33 &= 1 \cdot 18 + 15 \\ 18 &= 1 \cdot 15 + 3 \\ 15 &= 5 \cdot 3 + 0 \end{aligned}$$

so $\gcd(51, 288) = 3$.

(b):

$$\begin{aligned} 629 &= 1 \cdot 357 + 272 \\ 357 &= 1 \cdot 272 + 85 \\ 272 &= 3 \cdot 85 + 17 \\ 85 &= 5 \cdot 17 + 0 \end{aligned}$$

so $\gcd(357, 629) = 17$.

(c):

$$\begin{aligned} 252 &= 1 \cdot 180 + 72 \\ 180 &= 2 \cdot 72 + 36 \\ 72 &= 2 \cdot 36 + 0 \end{aligned}$$

so $\gcd(180, 252) = 36$.

Example 59. Prove that the square of every odd integer is of the form $4k+1$, where $k \in \mathbb{Z}$ (that is, for each odd integer $a \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $a^2 = 4k+1$).

Solution: Suppose that a is an odd integer, then there exists a $j \in \mathbb{Z}$ such that $a = 2j+1$. Thus we have

$$a^2 = (2j+1)^2 = 4j^2 + 4j + 1 = 4(j^2 + j) + 1.$$

Since $j \in \mathbb{Z}$, we also have $j^2 + j \in \mathbb{Z}$, so letting $k = j^2 + j$ proves the claim.

Example 60. Prove that if a divides b and c divides d, then ac divides bd.

Solution: If a divides b and c divides d, then there are integers j and k such that $b = aj$ and $d = ck$. Then we see that $bd = (ac)(jk)$, and jk is an integer because both j and k are, this shows that ac divides bd.

Example 61. Answer true or false and give a complete justification. If p is prime, then p^2+1 is prime.

Solution: False. 3 is a prime number, but $3^2 + 1 = 10$ is not prime (since it factors as $2 \cdot 5$).

Example 62. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid b$ then $\gcd(a, c) = 1$. (Hint: Use proof by contradiction)

Solution: Assume a, b, c are integers which satisfy $\gcd(a, b) = 1$ and $c \mid b$. Assume, by way of contradiction $d := \gcd(a, c) > 1$. Thus, $d \mid a$ and $d \mid c$, so there exist integers k, m such that $a = dk$ and $c = dm$. On the other had, $c \mid b$, so $b = cn$ for some $n \in \mathbb{Z}$. This implies $b = dmn$. I.e., $d \mid a$ and $d \mid b$, meaning d is a common divisor of a and b, so $1 = \gcd(a, b) \geq d > 1$. We have thus reached a contradiction.

Equivalence Relations and Modular Arithmetic

Example 63. For $(a, b), (c, d) \in \mathbb{R}^2$ define $(a, b) \sim (c, d)$ to mean that $2a - b = 2c - d$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

Solution: To prove $\sim$ is an equivalence relation on $\mathbb{R}^2$, we need to show it satisfies reflexivity, symmetry and transitivity.

1. *Reflexivity:* Let $(a, b) \in \mathbb{R}^2$ be an arbitrary element. We need to show $(a, b) \sim (a, b)$. Indeed, $2a - b = 2a - b$ for all $a, b \in \mathbb{R}$, thus $(a, b) \sim (a, b)$.

2. *Symmetry:* Let $(a,b) \in \mathbb{R}^2$ and $(c,d) \in \mathbb{R}^2$ be arbitrary elements. We need to show $(a,b) \sim (c,d)$ implies $(c,d) \sim (a,b)$. Indeed, if $2a - b = 2c - d$ then $2c - d = 2a - b$, thus $(a,b) \sim (c,d)$ implies $(c,d) \sim (a,b)$.

3. *Transitivity:* Let $(a,b), (c,d)$ and (e,f) be arbitrary elements in $\mathbb{R}^2$. We need to show that if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $(a,b) \sim (e,f)$. Indeed, assume $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. This means $2a - b = 2c - d$ and $2c - d = 2e - f$, which implies $2a - b = 2e - f$, i.e. $(a,b) \sim (e,f)$. Thus, $\sim$ is transitive.

Since $\sim$ satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation on $\mathbb{R}^2$.

Example 64. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $4 \mid (x + 3y)$. Prove $\sim$ is an equivalence relation. Describe its equivalence classes.

Solution: To prove $\sim$ is an equivalence relation on $\mathbb{Z}$, we need to show it satisfies reflexivity, symmetry and transitivity.

1. *Reflexivity:* Let $a \in \mathbb{Z}$ be an arbitrary element. We need to show $a \sim a$. Indeed, $4 \mid (a + 3a)$ for all $a \in \mathbb{Z}$, thus $a \sim a$ for all $a \in \mathbb{Z}$.

2. *Symmetry:* Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be arbitrary elements. We need to show $a \sim b$ implies $b \sim a$. Indeed, if $4 \mid (a + 3b)$ then there exists $k \in \mathbb{Z}$ such that $a + 3b = 4k$. Then $b + 3a = 4k - 2b + 2a$. Now, note that $a - b = 4k - 4b$, thus $b + 3a = 4k + 2(a - b) = 4k + 8(k - b) = 4(3k - 8b)$. Therefore, we can conclude that $4 \mid (b + 3a)$, since $(3k - 8b) \in \mathbb{Z}$, as k and b are integers. We have proven $a \sim b$ implies $b \sim a$, i.e., $\sim$ is symmetric.

3. *Transitivity:* Let a, b and c be arbitrary elements in $\mathbb{Z}$. We need to show that if $a \sim b$ and $b \sim c$, then $a \sim c$. Indeed, assume $a \sim b$ and $b \sim c$. This means $4 \mid (a + 3b)$ and $4 \mid (b + 3c)$. Then, there exist integers m, n such that $a + 3b = 4m$ and $b + 3c = 4n$. Adding these two equations we arrive at $a + 4b + 3c = 4m + 4n$. Thus, $a + 3c = 4(m + n - b)$. Since $(m + n - b) \in \mathbb{Z}$, we can conclude $4 \mid (a + 3c)$. Thus, $a \sim b$ and $b \sim c$ implies $a \sim c$.

Since $\sim$ satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation on $\mathbb{Z}$.
To describe the equivalence classes, note that $a \sim b$ is equivalent to $a + 3b = 4k$ for some $k \in \mathbb{Z}$, which is equivalent to $a - b = 4(k - b)$. Thus, if $a \sim b$, then $a \equiv b \pmod 4$. Conversely, if $a \equiv b \pmod 4$, i.e. $a - b = 4s$ for some $s \in \mathbb{Z}$, then $a + 3b = 4s + 4b$, thus $a \sim b$. We can conclude that $[a] = \{b \in \mathbb{Z} : b \equiv a (mod 4)\}$, i.e. we have four equivalence classes, which coincide with the equivalence classes in $\mathbb{Z}_4$, namely $[0] = \{..., -8, -4, 0, 4, 8, 12, ...\}$, $[1] = \{..., -7, -3, 1, 5, 9, 13, ...\}$, $[2] = \{..., -6, -2, 2, 6, 10, 14, ...\}$, and $[3] = \{..., -5, -1, 3, 7, 11, 15, ...\}$.

Example 65. Let $X = \mathbb{R}^2$, the xy-plane. Define $(x_1, y_1) \sim (x_2, y_2)$ to mean

$$x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Is $\sim$ an equivalence relation? Justify your answer. Give a geometric interpretation of the equivalence classes of $\sim$.

Solution: First, we will prove $\sim$ is an equivalence relation on $\mathbb{R}^2$. For this we need to show it satisfies reflexivity, symmetry and transitivity.

1. *Reflexivity:* Let $(x, y) \in \mathbb{R}^2$ be an arbitrary element. We need to show $(x, y) \sim (x, y)$. Indeed, $x^2 + y^2 = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$, thus $\sim$ is reflexive.

2. *Symmetry:* Let $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$ be arbitrary elements. We need to show $(x_1, y_1) \sim (x_2, y_2)$ implies $(x_2, y_2) \sim (x_1, y_1)$. Indeed, if $x_1^2 + y_1^2 = x_2^2 + y_2^2$ then $x_2^2 + y_2^2 = x_1^2 + y_1^2$. Thus $(x_1, y_1) \sim (x_2, y_2)$ implies $(x_2, y_2) \sim (x_1, y_1)$, i.e., $\sim$ is symmetric.

3. *Transitivity:* Let $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) be arbitrary elements in $\mathbb{R}^2$. We need to show that if $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, then $(x_1, y_1) \sim (x_3, y_3)$. Indeed, assume $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. This means $x_1^2 + y_1^2 = x_2^2 + y_2^2$ and $x_2^2 + y_2^2 = x_2^3 + y_3^2$, which implies $2x_1^2 + y_1^2 = x_3^2 + y_3^2$, i.e. $(x_1, y_1) \sim (x_3, y_3)$. Thus, $\sim$ is transitive.

Since $\sim$ satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation on $\mathbb{R}^2$.

Example 66. Do the following calculations in $\mathbb{Z}_9$ (see page 238 of the text for a description of this notation), in each case expressing your answer as $[a]$ with $0 \leq a \leq 8$.

$$(a)\ [8] + [8] \qquad (b)\ [24] + [11] \qquad (c)\ [21] \cdot [15] \qquad (d)\ [8] \cdot [8].$$

Solution:

(a) $[8] + [8] = [16] = [7]$.

(b) $[24] + [11] = [6] + [2] = [6 + 2] = [8]$.

(c) $[21] \cdot [15] = [3] \cdot [6] = [18] = [0]$.

(d) $[8] \cdot [8] = [-1] \cdot [-1] = [1]$.

Example 67. Let a and b be given integers. Prove $a \equiv b \mod 5$ if and only if $9a + b \equiv 0 \mod 5$.

Solution: Assume $a \equiv b \mod 5$. Then, $a - b = 5k$ for some integer k. Thus, $9a + b = 9a + (a - 5k) = 10a - 5k = 5(2a - k)$, Thus, $5 \mid (9a + b - 0)$, i.e., $9a + b \equiv 0 \mod 5$.
To prove the converse, assume $9a + b \equiv 0 \mod 5$. Then, there exists an integer s such that $9a + b = 5s$. Thus, $a - b = a - (5s - 9a) = 10a - 5s = 5(2a - s)$. Thus, $5 \mid (a - b)$, i.e., $a \equiv b \mod 5$.