CARDINALITY OF SETS

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1. Cardinality

Definition 1. Given sets A and B, we say that A and B *have the same cardinality*, written $A \sim B$, if there is a bijection $h : A \to B$.

Recall that a bijection is a map $h : A \to B$ that is

(1) *Injective*, also called one-to-one, which means that $h(x) = h(y)$ implies $x = y$; and
(2) *Surjective,* also called onto, which meanse that for every $z \in B$ there is $x \in A$ such that $h(x) = z$.

Proposition 1. *$\sim$ is an equivalence relation on the class of sets. That is for any sets A, B, C,*

(1) $A \sim A$.
(2) *If $A \sim B$ then $B \sim A$.*
(3) *If $A \sim B$ and $B \sim C$ then $A \sim C$.*

Remark 1. One can define equivalence relations on "proper classes," which are collections of objects too large to be sets (because of Russell's paradox). We can still speak about the *equivalence classes*; in this case the class of all sets with a given cardinality.

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Proof. To see reflexivity, use the identity map on A. To prove symmetry use the inverse map for the bijection $h : A \to B$. To prove transitivity use the composition $g \circ h$ wherer $h : A \to B$ and $g : B \to C$ are bijections. $\square$

Definition 2. The *cardinality* of a set A, denoted $\mathrm{card}(A)$, is the equivalence class of A.

Definition 3. Let $[0] = \emptyset$ and for each $n \in \mathbb{N}$ with $n > 0$ define $[n] = \{0, 1, \ldots, n-1\}$.

Definition 4. Let A be a set. We say that

(1) A is *finite* if $\mathrm{card}(A) = \mathrm{card}([n])$ for some $n \in \mathbb{N}$.
(2) A is *infinite* if it is not finite;
(3) A is *countable* if $\mathrm{card}(A) = \mathrm{card}(\mathbb{N})$;
(4) A is *uncountable* if A is infinite but not countable; and
(5) A is *almost countable* if A is finite or countable.

If A is finite and $\mathrm{card}(A) = \mathrm{card}([n])$ we write $\mathrm{card}(A) = n$; if A is countable we write $\mathrm{card}(A) = \aleph_0$. $\aleph$, pronounced "Aleph," is the first letter of the Hebrew alphabet.

2. Counting Finite Sets

Theorem 1. *Let $m, n \in \mathbb{N}$. If $m < n$ then there is no surjection from $[m]$ to $[n]$.*

Before proving this, note that it implies directly the

Corollary 1. *If $m, n \in \mathbb{N}$ and $m \neq n$, then $\mathrm{card}([n]) \neq \mathrm{card}([m])$.*

In particular, it makes sense to write $\mathrm{card}(A) = n$ when $\mathrm{card}(A) = \mathrm{card}([n])$.

Proof of Theorem. We will prove the contrapositive: *if there is a surjection from $[m]$ to $[n]$ then $n \leq m$.* This can be proved by induction on m.

For the base case, recall that $[0] = \emptyset$. Suppose there is a surjection $f : \emptyset \to [n]$. Thus $[n] = \{f(x) \ : \ x \in \emptyset\} = \emptyset$. So $n = 0$, and the base case holds.

Now suppose, for some m, that $n \leq m$ *whenever there is a surjection from $[m]$ to $[n]$*. Suppose we are given a surjective map $h : [m+1] \to [n]$. We must show $n \leq m+1$. Let $g : [m] \to [n]$ denote the restriction of h to $[m] \subset [m+1]$, i.e., $g(x) = h(x)$ for $x \in [m]$. If g is a surjection then $n \leq m < m+1$ and we are done. If, on the other hand, g is not a surjection, then there is $\alpha \in [n]$ such that $h(x) = g(x) \neq \alpha$ for every $x \in [m]$. Since h is a surjection and $[m+1] \setminus [m] = \{m\}$, it follows that $h(m) = \alpha$. Now let $f : [n] \to [n]$ be the map

$$f(k) = \begin{cases} k & \text{if k} \neq \alpha \text{ and } k \neq n-1, \\ n-1 & \text{if } k = \alpha, \text{ and} \\ \alpha & \text{if } k = n-1. \end{cases}$$

So f just interchanges α and $n-1$. Since $f \circ f$ is the identity map, it is easy to see that f is a bijection. Then $f \circ h$ has the properties that 1) $f \circ h(m) = n-1$, 2) $f \circ h$ is a surjection and 3) $f \circ h(x) = f \circ g(x) \neq n-1$ for $x \in [m]$. It follows that $f \circ g : [m] \to [n-1]$ is a surjection! So$n - 1 \leq m$ and thus $n \leq m$. □

That was painful, but notice that we have now *proved* that our intuitive idea of counting elements of a set works. And it all follows from induction.

Theorem 2. *Let A be a finite set. If $B \subset A$, then $\operatorname{card}(B) \leq \operatorname{card}(A)$ with equality only if $A = B$.*

Proof. First note that, since $A = B$ clearly implies $\operatorname{card}(B) = \operatorname{card}(A)$, the result is clearly equivalent to the statement: *If $B \subset A$ and $B \neq A$ then* $\operatorname{card}(A) < \operatorname{card}(B)$. It suffices to prove this for $A = [n]$ with $n \in \mathbb{N}$, for which we may use induction on n.

The base case is vacuously true, since there are no proper subsets of the emptyset.

Suppose that the result holds for $A = [n]$ and let B be a subset of $[n+1]$ with $B \neq [n+1]$. We consider two cases:

Case 1: $n \notin B$. Then $B \subset [n]$. Either $B = [n]$, in which case $\operatorname{card}(B) = n < n+1$; or $B \neq [n]$, in which case $\operatorname{card}(B) \leq n < n+1$ by the induction hypothesis.

Case 2: $n \in B$. Let $C = B \setminus \{n\}$. So, $C \subset [n]$ and $C \neq [n]$, since $B = C \cup \{n\} \neq [n+1]$. Thus $\operatorname{card}(C) < n$ by the induction hypothesis. That is, there is a bijection $h : C \to [m]$ where $m < n$. Define $g : B \to [m+1]$ by $g(n) = m$ and $g(x) = h(x)$ if $x \in C$. Then g is bijection — it is surjective since the image of h is $[m]$ and $g(n) = m$; and is injective since h is injective and $h(x) \neq m$ for $x \in C$. Thus $\operatorname{card}(B) = m + 1 < n + 1$.

□

Corollary 2 (Pigeon hole principle)**.** *Let* A*,* B *be finite sets. If* $\mathrm{card}(B) < \mathrm{card}(A)$*, then no map* $h : A \to B$ *is injective.*

Proof. Prove the contrapositive. Suppose there is a map $h : A \to B$ that is injective. Then $\mathrm{card}(A) = \mathrm{card}(h(A)) \leq \mathrm{card}(B)$ since $h(A) \subset B$. □

This is called the "Pigeon hole principle" for the following reason. Imagine that the elements of A are pigeons and the elements of B are holes in which the pigeons could roost. The function $h : A \to B$ is an assignment of pigeons to holes. The theorem simply says that, if there are more pigeons than holes, then some hole must contain more than one pigeon!

3. Ordering of Infinite cardinalities

More generally we can define an order relation on the collection of all cardinalities by

Definition 5. We say that $\mathrm{card}(A) \leq \mathrm{card}(B)$ if there is an injective map $h : A \to B$.

For finite cardinalities, this agrees with the usual notion by the contrapositive of the Pigeon hole principle.

Theorem 3 (Cantor-Schroeder-Bernstein)**.** *If* $\mathrm{card}(A) \leq \mathrm{card}(B)$ *and* $\mathrm{card}(B) \leq \mathrm{card}(A)$ *then* $\mathrm{card}(A) = \mathrm{card}(B)$.

That is, *if* $h : A \to B$ *and* $g : B \to A$ *are injections then there exists a bijection* $f : A \to B$. If A is finite, this result is easy. Indeed, it that case $g \circ h$ is an injection from A to A, and so must be surjective by the Pigeon hole principle. Thus g must be surjective, so g is a bijection. Likewise h is a bijection. On the other hand, if A is infinite then h need not be a bijection as the following example shows:

Example 1 (Hilbert's hotel)**.** Hilbert's hotel has infinitely many rooms indexed by the natural numbers. This amazing hotel need never turn away a guest. Indeed, if the hotel is full and a guest shows up looking for a room, the manager simply asks each guest to move to the next room, thereby freeing up room 0 for the new guest.

Put in mathematical terms, Hilbert's hotel works because the map $h : \mathbb{N} \to \mathbb{N}\setminus\{0\}$ given by $h(n) = n+1$ is a bijection. Thus $\mathrm{card}(\mathbb{N}) = \mathrm{card}\left(\mathbb{N} \setminus \{0\}\right)$. Note that in this case we also have a injection $g : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ given by $g(n) = n$ and that $g \circ h$ is ***NOT SURJECTIVE!***. This is why we need the Cantor-Schroeder-Bernstein theorem. You can read its proof, for example, on Wikipedia.

Definition 6. We say that $\mathrm{card}(A) < \mathrm{card}(B)$ if $\mathrm{card}(A) \leq \mathrm{card}(B)$ but $\mathrm{card}(A) \neq \mathrm{card}(B)$.

By Cantor-Schroeder-Bernstein, this is equivalent to $\mathrm{card}(A) \leq \mathrm{card}(B)$ but not $\mathrm{card}(B) \leq \mathrm{card}(A)$.

Corollary 3. *The relation $<$ is a total order on cardinalities. That is, it is transtive and for any A, B exactly one of $\mathrm{card}(A) < \mathrm{card}(B)$, $\mathrm{card}(A) = \mathrm{card}(B)$ or $\mathrm{card}(B) < \mathrm{card}(A)$ holds.*

Proof. Transitivity is easy, since if $h : A \to B$ and $g : B \to C$ are injections, then $g \circ h$ is an injection. By Cantor-Schroeder-Bernstein, only one of the cases $\mathrm{card}(A) < \mathrm{card}(B)$, $\mathrm{card}(A) = \mathrm{card}(B)$ or $\mathrm{card}(B) < \mathrm{card}(A)$ can hold. The proof that one of the *does* hold requires the axiom of choice and goes beyond what we need to get into in this course. □

Theorem 4 (Generalized pigeon hole principle)**.** *Let A, B be sets. If $\mathrm{card}(B) < \mathrm{card}(A)$, then no map $h : A \to B$ is injective.*

Proof. We prove the contrapositive. Suppose $h : A \to B$ is injective. Then $\mathrm{card}(A) \leq \mathrm{card}(B)$, which is the negation of $\mathrm{card}(B) < \mathrm{card}(A)$ by the fact that $<$ is a total order. □

Theorem 5. *If $A \subset B$, A is finite and $A \neq B$ then $\mathrm{card}(A) < \mathrm{card}(B)$.*

Proof. Since $A \subset B$ we have $\mathrm{card}(A) \leq \mathrm{card}(B)$ (because the identity map is an injection). Since $A \neq B$ there is $x \in B \setminus A$. Since A is finite we have

$$\mathrm{card}(A) < \mathrm{card}(A) + 1 = \mathrm{card}(A \cup \{x\}) \leq \mathrm{card}(B).$$ □

This result fails catastrophically if A is infinite, as the example of Hilbert's hotel shows: $\mathrm{card}(\mathbb{N}) = \mathrm{card}\left(\mathbb{N} \setminus \{0\}\right)$. Actually, we even have

$$\mathrm{card}(\mathbb{N}) = \mathrm{card}\left(\mathbb{Z}\right) = \mathrm{card}\left(\mathbb{Q}\right).$$

However, $\operatorname{card}(\mathbb{N}) < \operatorname{card}(\mathbb{R})$, as we will show.

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