COMPACTNESS IN METRIC AND TOPOLOGICAL SPACES

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1. Limit points and sequential limits

Recall that a point p is a limit point of a set E in a metric space if every open ball $B_r(p)$ centered at p contains a point of E not equal to p.

Lemma 1. *Let E be a subset of a metric space X. A point p is a limit point of E if and only if there is a sequence $(x_n)_{n=1}^\infty$ in E such that $x_n \neq p$ for all n and $\lim_n x_n = p$.*

Proof. If p is a limit point then choose for each n a point $x_n \in B_{\frac{1}{n}}(p)$ with $x_n \neq p$. Since $d(x_n, p) < \frac{1}{n} \to 0$ we see that $\lim_n x_n = p$. In the other direction suppose that $x_n \to p$ with $x_n \in E$ and $x_n \neq p$ for all n. Then given $r > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_r(p)$ — this is the definition of convergence! So $B_r(p)$ actually contains infinitely many points not equal to p. $\square$

Definition 1. Let X be a metric space and let $(x_n)_{n=0}^\infty$ be a sequence in X. A *subsequence* of $(x_n)_{n=1}^\infty$ is a sequence $y_k = x_{n_k}$ for $k = 0, 1, \ldots$ where n_k is an increasing sequence of natural numbers.

Recall that a sequence $(x_n)_{n=1}^\infty$ is *Cauchy* if for every $\epsilon > 0$ there is N such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$ and that convergent sequences are Cauchy.

Lemma 2. *Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in a metric space. If $\lim_k x_{n_k} = x$ for some subsequence $(x_{n_k})_{k=1}^\infty$, then $\lim_n x_n = x$.*

Proof. Let $\epsilon > 0$. Let K be such that $k \geq K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$ and let N' be such that $n, m \geq N'$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$. Let $N = \max(N', n_K)$. Then for $n \geq N$ we have

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \epsilon.$$

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□

2. Complete Metric Spaces

Definition 2. A metric space X is *complete* if every Cauchy sequence in X converges in X.

We proved in class that $\mathbb{R}$ is complete. However, not all metric spaces are complete. For example, the space of rational numbers $X = \mathbb{Q}$, with the usual metric, is not complete since there are Cauchy sequences in $\mathbb{Q}$ that do not converge *to a point in* $\mathbb{Q}$. An even simpler example is $X = (0, 1)$, with the usual metric. The sequence $1/n$ is Cauchy, but has no limit in X. In each of these examples the metric space X sits naturally in a larger space and every Cauchy sequence in X converges in the larger space. This is no accident as we will now see.

Definition 3. Let X and Y be metric spaces. A map $T : X \to Y$ is an *isometry* if for any $x, y \in X$,

$$d_Y(T(x), T(y)) = d_X(x, y),$$

where d_X is the metric on X and d_Y is the metric on Y.

Note that an isometry is necessarily one-to-one. Indeed, if $T(x) = T(y)$ then $d_X(x, y) = 0$, so $x = y$. However an isometry need not be surjective. A surjective isometry is called an *iosmetric isomorphism* of metric spaces. We say that metric spaces X and Y are *isometric* if there is a surjective isometry from X to Y. Note that if T is a surjective isometry, then the inverse map is also an isometry. Also the composition of isometries is an isometry. It follows that isometric isomrophism is an equivalence relation on the class of metric spaces.

Theorem 1. *Let X be a metric space. Then there is a complete metric space Y and an isometry $T : X \to Y$.*

Proof. The idea is to construct Y from X by insisting that Cauchy sequences converge. So we will associate to each Cauchy sequence in X a point in Y. However, distinct Cauchy sequences can represent the same point, since the sequences may "converge to one another." So we start with an equivalence relation on Cauchy sequences. Let

$$\mathcal{C} \;=\; \{(x_n)_{n=1}^{\infty} \;:\; x_n \in X \text{ and } (x_n)_{n=1}^{\infty} \text{ is Cauchy.}\}$$

Define an equivalence relation $\mathcal{C}$ as follows

$$(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \quad \text{if and only if} \quad \lim_{n\to\infty} d_X(x_n, y_n) = 1,$$

where d_X is the metric on X.

Exercise 1. Prove that this is an equivalence relation.

Now let

$$Y \;=\; {}^{\mathcal{C}}\!/\!_{\sim} \;=\; \{\text{Equivalence classes of Cauchy sequences modulo } \sim .\}$$

We define a metric on Y by

$$d_Y\left(\left[(x_n)_{n=1}^{\infty}\right], \left[(y_n)_{n=1}^{\infty}\right]\right) \;:=\; \lim_{n\to\infty} d_X(x_n, y_n). \tag{2.1}$$

To do this we need the following

Lemma 3. *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in a metric space X. Then $\lim_n d_X(x_n, y_n)$ exists.*

Proof. Since $\mathbb{R}$ is complete, it suffices to prove that $(d_X(x_{n,}, y_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$ and let N be sufficiently large that

$$d_X\left(x_n, x_m\right) < \frac{\epsilon}{2} \quad \text{and} \quad d_X\left(y_n, y_m\right) < \frac{\epsilon}{2}$$

whenever $n, m > N$. By the triangle inequality, for $n, m > N$ we have

$$d_X\left(x_n, y_n\right) \leq d_X(x_n, x_m) + d_X(x_m, y_m) + d_X(y_m, y_n) < \epsilon + d_X(x_m, y_m)$$

and also the same inequality with n and m interchanged. Thus, for $n, m > N$,

$$-\epsilon \;<\; d_X\left(x_n, y_n\right) - d_X(x_m, y_m) \;<\; \epsilon,$$

which is $|d_X\left(x_n, y_n\right) - d_X(x_m, y_m)| < \epsilon$. Hence $(d_X(x_{n,}, y_n))_{n=1}^{\infty}$ is Cauchy. $\square$

Returning to the proof of the theorem, we now see that the right hand side of eq. (2.1) makes sense. However we still need to prove that d_Y is well defined, i.e., that the right hand side of eq. (2.1) does not depend on the choice of representives of the equivalence classes.

Exercise 2. Prove that d_Y is well defined by eq. (2.1) and that Y is complete.

It remains to construct the isometry T. This is done by mapping a point $x \in X$ to the Cauchy sequence $(x, x, x, \ldots)$. Let $(x)_{n=1}^{\infty}$ denote the sequence equal to x for each n and define

$$T(x) \ := \ [(x)_{n=1}^{\infty}] \, .$$

Clearly T is an isometry since

$$d_Y(T(x), T(y)) \ = \ \lim_{n \to \infty} d_X(x, y) \ = \ d_X(x, y). \qquad \square$$

Definition 4. Let X be a metric space. A *completion* of X is the closure of $T(X)$ for any isometry $T : X \to Y$ with Y a complete metric space.

Theorem 2. *Let Y_1 and Y_2 be complete metric spaces and let $T_j : X \to Y_j$ be an isometry for $j = 1, 2$. Then $\overline{T_1(X)}$ and $\overline{T_2(X)}$ are isometric.*

Proof. An element $y \in \overline{T_1(X)}$ is a limit of a sequence $T_1(x_n)$ with $x_n \in X$. Since T_1 is an isometry, we find that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since T_2 is an isometry, $(T_2\,(x_n))_{n=1}^{\infty}$ is Cauchy in Y_2. Since Y_2 is complete, the sequence has a limit. Call the limit $T(y)$. Since

$$T(y) = \lim_{n \to \infty} T_2\,(x_n)\,,$$

we have $T(y) \in \overline{T_2\,(X)}$. Furthermore, by reversing the above argument we see that any point $z \in \overline{T_2\,(X)}$ is equal to $T(y)$ for some $y \in \overline{T_1\,(X)}$. The map $T : \overline{T_1\,(X)} \to \overline{T_2\,(X)}$ is easily seen to be an isometry. Since it is also surjective, it is an isometric isomrophism. $\square$

Since any two completions are ismorphic we will typically speak of "the completion of X." Typically, we will think of the completion of X as the metric space $\widetilde{X}$ obtained by "adding to X all the limits of Cauchy sequences in X." At a formal level, $\widetilde{X}$ is the space constructed in the proof of Theorem 1 and we *identify* X with its image under the isometry constructed there so that $X \subset \widetilde{X}$. The completion $\widetilde{X}$ has the property that

> *If $T : X \to Y$ is an isometry, then there is a unique isometry $\widetilde{T} : \widetilde{X} \to Y$ such that $T(x) = \widetilde{T}(x)$ for $x \in X$.*

3. Notions of Compactness

Definition 5. Let X be a metric space and let $K \subset X$ be a subset.

(1) K is *limit point compact* if every infinite subset of K has a limit point in K.
(2) K is *sequentially compact* if every sequence $(x_n)_{n=0}^{\infty}$ in K has a subsequence that converges to a point of K.
(3) K is *bounded* if $K \subset B_R(x)$ for some $R > 0$ and some $x \in X$.
(4) K is *totally bounded* if for every $\epsilon > 0$ there are finitely many points $x_1, \ldots, x_n \in K$ such that
$$K \subset \bigcup_{j=1}^{n} B_\epsilon(x).$$

Theorem 3. *Let X be a metric space and $K \subset X$. The following are equivalent:*

(1) *K is compact,*
(2) *K is limit point compact,*
(3) *K is sequentially compact, and*
(4) *K is complete and totally bounded.*

In the proof we will use the following two lemmas:

Lemma 4 (Lebesgue number lemma)**.** *Let K be sequentially compact. If $\mathcal{G}$ is an open cover of K then there is a $\delta > 0$ such that for any $x \in K$ there is $U \in \mathcal{G}$ with $B_\delta(x) \subset U$.*

Remark 1. The point is that we can choose δ *uniformly*, which is to say independent of x. Since $\mathcal{G}$ is an open cover it is clear that
$$\forall x \in K\ \exists \delta > 0\ \exists U \in \mathcal{G},\ B_\delta(x) \subset U.$$
However, the Lemma asserts that
$$\exists \delta > 0\ \forall x \in K\ \exists U \in \mathcal{G},\ B_\delta(x) \subset U.$$
This interchange of quantifiers is not universally valid – it depends on the properties of metric spaces.

Proof. We prove the contrapositive: if $\mathcal{G}$ is a collection of open sets such that
$$\forall \delta > 0\ \exists x \in K\ \forall U \in \mathcal{G},\ B_\delta(x) \not\subset U$$

then $\mathcal{G}$ is not a cover of K. Thus, suppose $\mathcal{G}$ is a collection of open sets and for each $n \in \mathbb{N}_+ \subset \{n \in \mathbb{N} : n > 0\}$ there is a point $x_n \in K$ such that $B_{\frac{1}{n}}(x_n)$ is not contained in any open set of the collection $\mathcal{G}$. Let $(x_{n_k})_{k=0}^{\infty}$ be a subsequence converging to some point x. Let $\delta > 0$. Then for large enough k, $x_{n_k} \in B_{\frac{\delta}{2}}(x)$ and $\frac{1}{n_k} < \frac{\delta}{2}$. It follows from the triangle inequality that $B_{\frac{1}{n_k}}(x_{n_k}) \subset B_\delta(x)$ so that $B_\delta(x) \not\subset U$ for any $U \in \mathcal{G}$. Thus for every $U \in \mathcal{G}$ and for every $\delta > 0$, $B_\delta(x) \not\subset U$. Since the sets in $\mathcal{G}$ are open, it follows that $x \notin U$ for any $U \in \mathcal{G}$. $\square$

Proof of Theorem 3. We will prove 2$\Longrightarrow$3, 3$\Longrightarrow$4, 4$\Longrightarrow$3 and 3$\Longrightarrow$1. The implication 1 $\Longrightarrow$ 2 is Theorem 2.37 in Rudin.

To prove 2$\Longrightarrow$3, suppose K is limit point compact. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in K. Consider the set $E = \{x_n : n = 1, 2, \ldots\}$. If E is finite then there is $p \in E$ such that $x_n = p$ for infinitely many n, so there is a subsequence (x_{n_k}) with $x_{n_k} = p$ for all k. Clearly $x_{n_k} \to p$. If E is infinite then it has a limit point p in K by assumption. Thus every neighborhood $B_r(p)$ of p contains infinitely many points of E. Let n_1 such that $x_{n_1} \in B_1(p) \cap E$ and $x_{n_1} \neq p$. For each $k > 1$ choose n_k such that $n_k > n_{k-1}$, $x_{n_k} \in B_{\frac{1}{k}}(p)$ and $x_{n_k} \neq p$. (We may insist that $n_k > n_{k-1}$ since there are infinitely many points of the sequence in $B_{\frac{1}{k}}(p)$.) Since $d(x_{n_k}, p) = \frac{1}{k} \to 0$ we see that $x_{n_k} \to p$.

Let us now prove that 3$\Longrightarrow$4. We argue by contrapositive. Suppose K is not complete or not totally bounded. If K is not complete then there is a Cauchy sequence in K that does not converge. By Lemma 2 this sequence can have no convergent subsequence so K is not sequentially compact. On the other hand if K is not totally bounded then there is some $\epsilon > 0$ such that no finite collection of ϵ balls with centers in K covers K. So we can find a sequence $x_1, x_2, \ldots$ in K such that $x_k \notin B_\epsilon(x_j)$ for $k \neq j$. Since $d(x_k, x_j) \geq \epsilon$ for $k \neq j$ we see that no subsequence can possibly be Cauchy so no subsequence can converge. Thus K is not sequentially compact.

To prove 4$\Longrightarrow$3, suppose that K is complete and totally bounded and let $(x_n)_{n=1}^{\infty}$ be a sequence in K. We will define a *sequence of subsequences* of (x_n) as follows:

(1) Let $B_1(p_1), \ldots, B_1(p_n)$ be finitely many balls of radius 1 that cover K. Let $\left(x_{n_k^{(1)}}\right)_{k=1}^{\infty}$ be a subsequence that is entirely contained in one of

these balls. This exists since one of the balls must contain infinitely many elements of the sequence.

(2) Given the subsequence $\left(x_{n_k^{(m)}}\right)_{k=1}^{\infty}$ let $B_{\frac{1}{m+1}}\left(p_1\right), \ldots, B_{\frac{1}{m+1}}\left(p_n\right)$ be finitely many $\frac{1}{m+1}$ balls that cover K and let $\left(x_{n_k^{(m+1)}}\right)_{k=1}^{\infty}$ be a *subsequence of* $\left(x_{n_k^{(m)}}\right)_{k=1}^{\infty}$ that is entirely contained in one of these balls.

Note that the subsequences produced have the following properties:

(1) The points of $\left(x_{n_k^{(m)}}\right)_{k=1}^{\infty}$ are entirely contained in a ball of radius $\frac{1}{m}$ so $d(x_{n_k^{(m)}}, x_{n_\ell^{(m)}}) < \frac{2}{m}$ for each k, ℓ.

(2) Each sequence $\left(x_{n_k^{(m)}}\right)_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$ and, for $m \geq 2$, a subsequence of $\left(x_{n_k^{(m-1)}}\right)_{k=1}^{\infty}$.

Now consider the sequence $x_{n_k} = x_{n_k^{(k)}}$. That is, x_{n_k} is the first element of the k^{th} subsequence constructed. Note that if $j \geq m$ then x_{n_j} appears in the sequence $\left(x_{n_k^{(m)}}\right)_{k=0}^{\infty}$ so we have

$$d(x_{n_j}, x_{n_m}) < \frac{1}{m}.$$

Thus the sequence $(x_{n_k})_{k=}^{\infty}$ is Cauchy! Since K is complete it has a limit. Since the sequence (x_n) was arbitrary we see that K is sequentially compact.

Finally let us prove that 3$\Longrightarrow$1. To this end suppose K is sequentially compact and let $\mathcal{G}$ be an open cover of K. By Lebesgue's number lemma there is a $\delta > 0$ such that for each $x \in K$ we have $U \in \mathcal{G}$ with $B_\delta\left(x\right) \subset U$. Since we have shown that sequential compactness implies totally bounded we know that there are points $x_1, \ldots, x_n$ such that $\bigcup_{j=1}^{n} B_\delta\left(x_j\right)$ covers K. Let $U_1, \ldots, U_n$ be in $\mathcal{G}$ such that $B_\delta\left(x_j\right) \subset U_j$. Clearly $K \subset \bigcup_j U_j$ so we have found a finite subcover. As $\mathcal{G}$ was an arbitrary open cover, we see that K is compact. □

Corollary 1. *A subset K of a complete metric space is compact if and only if it is closed and totally bounded.*

Corollary 2. *A subset $E \subset \mathbb{R}^k$ is totally bounded if and only if E is bounded.*

This will follow once we prove the Heine-Borel theorem that any closed and bounded subset of $\mathbb{R}^k$ is sequentially compact. There are, however, metric spaces in which bounded is distinct from totally bounded. A somewhat silly example of one is any infinite set X with the discrete metric

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Clearly X is bounded, since $X \subset B_{1+\epsilon}(x)$ for any point $x \in X$. However, X is not totally bounded since it is infinite and $B_r(x) = \{x\}$ if $r < 1$. We will see more natural examples of such metric spaces later on.

Exercise 3. What are the compact sets of an infinite set with the discrete metric?

4. Finite intersection property for closed sets

Definition 6. Let $\mathcal{F}$ be a collection of subsets of a metric space. We say that $\mathcal{F}$ has the *finite intersection property* if for every finite collection $F_1, \ldots, F_n \in \mathcal{F}$ we have

$$F_1 \cap \ldots \cap F_n \neq \emptyset.$$

Theorem. *A subset K of a metric space X is compact if and only if it is closed and any collection $\mathcal{F}$ of closed subsets of K with the finite intersection property satisfies*

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$

Proof. This looks complicated, but it is just the definition of compactness rephrased in terms of open sets!

First suppose X is compact. Then X is closed and given a collection of closed subsets $\mathcal{F}$ with the FIP we see that

$$F_1^c \cup \ldots \cup F_n^c \not\supset K$$

for any finite family $F_1, \ldots, F_n \in \mathcal{F}$. Since the sets F^c are open it follows that $\{F^c : F \in \mathcal{F}\}$ is not a cover of K. Thus there is a point $p \in K$ such that

$$p \in \left(\bigcup_{F \in \mathcal{F}} F^c \right)^c = \bigcap_{F \in \mathcal{F}} F.$$

So $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.

Now suppose conversely that K is closed and any collection of closed subsets with the FIP has non-empty intersection and let $\mathcal{G}$ be an open cover of K. Since

$$K \subset \bigcup_{U\in\mathcal{G}} U$$

we conclude that

$$\emptyset = K \cap \left(\bigcup_{U\in\mathcal{G}} U\right)^c = \bigcap_{U\in\mathcal{G}} \left(K \cap U^c\right).$$

Since $\{K \cap U^c \ : \ U \in \mathcal{G}\}$ is a collection of closed subsets of K we conclude that it does not have the FIP. Thus there is a finite collection $K \cap U_1^c, \ldots, K \cap U_n^c$ such that

$$\emptyset = \bigcap_{j=1}^{n} K \cap U_j^c = K \cap \left(\bigcup_{j=1}^{n} U_j\right)^c.$$

It follows that $K \subset \bigcup_{j=1}^n U_j$ so $\mathcal{G}$ has a finite subcover. □

One extremely important corollary of this is the following result

Corollary 3. *Let $K_1 \supset K_2 \supset \cdots$ be a decreasing sequence of compact subsets of a metric space X. Then $\bigcap_n K_n \neq \emptyset$. If furthermore $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$ then*

$$\bigcap_n K_n = \{p\}$$

for some point $p \in X$.

Here

$$\operatorname{diam}(S) = \sup\{d(x,y) \ : \ x, y \in S\}$$

is the *diameter* of a subset of a metric space.

5. Topological spaces

The above proof used some very special properties of metric spaces, which are the only sort of spaces we will deal with in this course. However there is a more general notion of a *topological space* defined as follows.

Definition 7. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

(1) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$,
(2) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$,
(3) If $\mathcal{G} \subset \mathcal{T}$ then $\displaystyle\bigcup_{U \in \mathcal{G}} U \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* and a space endowed with a topology is called a *topological space.*

Note that the open sets in a metric space satisfy 1,2, and 3. So a metric space is a special kind of topological space. However, there do exist topological spaces for which no metric can be used to define the topology. One can define notions like convergence and compactness in the context of topological spaces:

Definition 8. Let X be a topological space. Then

(1) A point $p \in X$ is called a *limit point* of a set $E \subset X$ if every open set U containing p contains a point of E not equal to p.
(2) A sequence $(x_n)_{n=0}^{\infty}$ in X converges to a point $x \in X$ if for every open set U containing x there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have $x_n \in U$.
(3) A subset $K \subset X$ is called *compact* if every covering of K by open sets has a finite subcover.
(4) A subset K is called *limit point compact* if every infinite subset of K has a limit point in K.
(5) A subset K is called *sequentially compact* if every sequence in K has a convergent subsequence.

Theorem 4. *Compact implies limit point compact, and sequential compact implies limit point compact. However, the converse implications can fail in a topological space without a metric. There are topological spaces that are compact but not sequentially compact, as well as spaces that are sequentially compact but not compact.*

The proofs that compact implies limit point compact and that sequentially compact implies limit point compact proceed much as in the metric space case. However the counter examples needed to show the other parts of the theorem are not really elementary.