Name: ____________________

Section: __________ **(Recitation) Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions may have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Compute the derivatives of the following functions: (**DO NOT SIMPLIFY**)

   (a) (4 points) $f(x) = x\sin(x) + \sqrt{x}$

   Solution:

   $$f'(x) = (1)\sin(x) + (x)\cos(x) + \frac{1}{2\sqrt{x}} = \boxed{\sin(x) + x\cos(x) + \frac{1}{2\sqrt{x}}}$$

   (b) (4 points) $g(x) = \dfrac{\tan x}{x^4 + 16}$

   Solution:

   $$g'(x) = \frac{\sec^2(x)(x^4+16) - \tan(x)(4x^3)}{(x^4+16)^2}$$

2. (6 points) Find the value of a that makes the function continuous at $x = 0$:

$$f(x) = \begin{cases} \dfrac{\sin(-5x)}{x} & \text{if } x < 0 \\ x - a & \text{if } x \geq 0 \end{cases}$$

Solution:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin(-5x)}{x} \\ &= \lim_{x\to 0^-} \frac{\sin(-5x)}{-5x} \cdot (-5) = -5 \end{aligned}$$

$f(0) = 0 - a = -a$, so the only way f can be continuous at $x = 0$ is if $\boxed{a = 5}$.

3. (7 points) The radius of a sphere is increasing at a rate of 3 mm/s. How fast is the volume increasing when the radius is 10 mm? (*Include units.*)

Solution:

$$V = \tfrac{4}{3}\pi r^3$$
$$V' = 4\pi(r^2)\cdot(r')$$
$$V' = 4\pi(100)\cdot(3) \qquad \text{(when } r = 10 \text{ and } r' = 3\text{)}$$

Giving us the final answer of $\boxed{1200\pi \text{ mm}^3/\text{s}}$.

4. (7 points) Using the Intermediate Value Theorem prove that the following equation has a root

$$x^4 + 1 = 3x^3.$$

Solution: Rearrange the equation to make $x^4 - 3x^3 + 1 = 0$ and define $f(x) = x^4 - 3x^3 + 1$ which we can apply the Intermediate Value Theorem on to show that $f(x) = 0$ since it is continuous everywhere (it's a polynomial).

$$f(0) = 1 > 0$$
$$f(1) = -1 < 0$$

So by the IVT there is some $c \in (0, 1)$ where $f(c) = 0$.

5. Calculate the following limits or show that they do not exist:

(a) (4 points) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} =$

Solution:

$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) \\ &= \lim_{x\to 9} \frac{x-9}{x-9} \cdot \left(\frac{1}{\sqrt{x}+3}\right) \\ &= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \boxed{\frac{1}{6}}\end{aligned}$$

(b) (4 points) $\lim_{x\to 2^-} \frac{x^3(x-2)}{|x-2|} =$

Solution: Since $x \to 2^-$ we know that $x < 2$ and so $|x-2| = -(x-2)$ giving us

$$\begin{aligned}\lim_{x\to 2^-} \frac{x^3(x-2)}{|x-2|} &= \lim_{x\to 2^-} \frac{x^3(x-2)}{-(x-2)} \\ &= \lim_{x\to 2^-} \frac{x^3}{-1} = \boxed{-8}\end{aligned}$$

6. (6 points) Consider the function $f(x) = \frac{1}{x+1}$. Use the limit definition of the derivative to compute f'.

Solution: $f(x+h) = \frac{1}{x+h+1}$ so

$$f(x+h) - f(x) = \frac{1}{x+h+1} - \frac{1}{x+1} = \frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)} = \frac{-h}{(x+1)(x+h+1)}$$

Therefore

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} = \boxed{\frac{-1}{(x+1)^2}}$$

7. The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h(t) = 2 + 24.5t - 4.9t^2$ after t seconds.

   (a) (4 points) Find the velocity at time t.

   Solution: $\boxed{v(t) = 24.5 - 9.8t}$

   (b) (3 points) When does the projectile reach it maximum height?

   Solution:

$$\begin{aligned} v(t) &= 0 \\ 24.5 - 9.8t &= 0 \\ 24.5 &= 9.8t \\ \boxed{\frac{24,5}{9.8}\text{ s}} &= t \end{aligned}$$

8. (7 points) Find the tangent line to the curve defined by the equation $xy^3 + \frac{2x}{y} = 9$ at the point $(1, 2)$.

Solution:

$$\begin{aligned} xy^3 + \frac{2x}{y} &= 9 \\ y^3 + x(3y^2y') + \frac{2y - 2xy'}{y^2} &= 0 \\ 8 + 12y' + \frac{4 - 2y'}{4} &= 0 \qquad \text{(when } x = 1 \text{ and } y = 2) \\ 8 + 12y' + 1 - \frac{1}{2}y' &= 0 \\ \frac{23}{2}y' &= -9 \\ y' &= \boxed{\frac{-18}{23}} \end{aligned}$$

Giving us the final result of $\boxed{y - 2 = \frac{-18}{23}(x - 1)}$

Multiple Choice. Circle the single best answer. No work needed. No partial credit available.

9. (4 points) Given $f(x) = |x - 7|$, which of the following statements is true?

   A. $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

   B. $f(x)$ is continuous on $(-\infty, \infty)$, and differentiable on $(-\infty, 7) \cup (7, \infty)$.

   C. $f(x)$ is differentiable on $(-\infty, \infty)$, and continuous on $(-\infty, 7) \cup (7, \infty)$.

   D. $f(x)$ is differentiable on $(-\infty, \infty)$, but not continuous at $x = 7$.

   E. $f(x)$ is not defined at $x = 7$.

10. (4 points) Suppose $f(x)$ is continuous and differentiable and that $f'(x) > 0$ always and $f(0) = 3$. What is true about $f(1)$?

    A. It is possible that $f(1) = 3$

    B. It must be that $f(1) < 3$

    C. It must be that $f(1) > 3$

    D. There is not enough information

11. (4 points) Which of the following equations are vertical asymptotes of $g(x) = \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$?

    A. $x = 1$

    B. $x = 3$

    C. $x = 1$ and $x = 3$

    D. $x = -3$

    E. None of the above

12. (4 points) Suppose

$$f(0) = 2,\ g(0) = 1,\ f'(0) = 6,\ g'(0) = 11.$$

Then the derivative of $\frac{f(x)}{g(x)}$ at $x = 0$ is

A. -16

B. 16

C. 28

D. -28

E. None of the above.

13. (4 points) Calculate the derivative of $f(x) = \cos(\tan x)$

A. $f'(x) = -\sin(\sec^2 x)$

B. $f'(x) = \sin(\tan x)\sec^2 x$

C. $f'(x) = -\sin(\tan x)\sec^2 x$

D. $f'(x) = \cos(\sec^2 x)$

E. $f'(x) = -\sin x \sec^2 x$

14. (4 points) Determine the average rate of change of the function $f(t) = 5 + \cos t$ on the interval $[0, \frac{\pi}{6}]$.

A. $\frac{6}{\pi}\left(\frac{\sqrt{3}}{2} + 5\right)$

B. $\frac{6}{\pi}\left(\frac{\sqrt{3}}{2} - 1\right)$

C. $\frac{6}{\pi}\left(\frac{\sqrt{1}}{2} + 5\right)$

D. $\frac{6}{\pi}\left(\frac{\sqrt{1}}{2}\right)$

E. $\frac{6}{\pi}\left(-\frac{\sqrt{1}}{2}\right)$

15. (4 points) Evaluate: $\lim_{x \to 1} \dfrac{\sin(x-1)}{x^2-1}$

A. -1

B. $\frac{1}{2}$

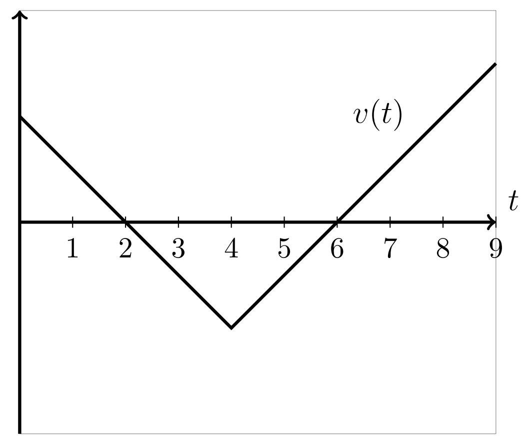
C. 2

D. 0

E. Does not exist

The following graph shows **the velocity** of a particle moving in a straight line for $t \in (0, 9)$.
Use it to answer the following two questions

16. (4 points) For what values of t is the particle moving forward?



A. $(4, 9)$

B. $(0, 2) \cup (6, 9)$

C. $(2, 6)$

D. $(0, 4)$

E. $(3, 5)$

17. (4 points) Which of the following statements is true of the particle at $t = 5$?

A. It is moving forwards and slowing down.

B. It is moving forwards and speeding up.

C. It is moving backwards and slowing down.

D. It is moving backwards and speeding up.

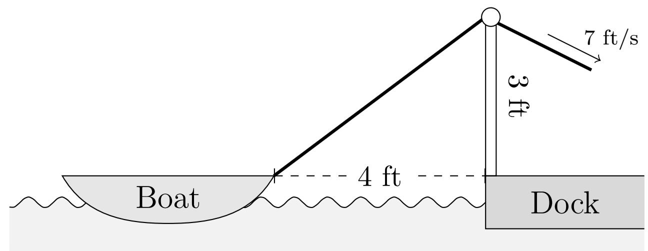
E. None of the above.

More Challenging Question(s).

18. (4 points) State whether the following is **True** or **False.** No work needed. $\dfrac{d}{dx}(\pi^4) = 4\pi^3$

Solution: FALSE. $\dfrac{d}{dx}(\pi^4) = 0$ since it is a constant.

19. (10 points) A boat is pulled into a dock by a rope attached to the bow (front end) of the boat and passing through a pulley on the dock that is 3 ft higher than the bow of the boat. If the rope is pulled in at a rate of 7 ft/s, at what speed is the boat approaching the dock when it is 4 ft from the dock?



Solution: Consider the following definitions

$$h(t) = \text{distance from the bow to the pully}$$
$$y(t) = \text{distance from the dock to the pully}$$
$$x(t) = \text{distance from the bow to the dock}$$

Then based on the picture and words above we have the following are true

$$h^2(t) = x^2(t) + y^2(t) \qquad h'(t) = -7 \qquad y(t) = 3$$

And there is a some time (lets say $t = a$) where the boat is 4 feet from the dock so we can write:

$$x(a) = 4$$

And our goal is to find $x'(a)$. Applying a derivative to our Pythagorean theorem equation we get

$$\begin{aligned} 2h(t)h'(t) &= 2x(t)x'(t) \\ h(t)(-7) &= x(t)x'(t) \\ h(a)(-7) &= (4)x'(a) \\ (5)(-7) &= (4)x'(a) \qquad \text{(using the Pythagorean equation again)} \\ \boxed{\frac{-35}{4}\text{ ft/s}} &= x'(a) \end{aligned}$$

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Algebraic

- $a^2 - b^2 = (a-b)(a+b)$
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- Quadratic Formula: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Geometric

- Area of Circle: πr^2
- Circumference of Circle: $2\pi r$
- Circle with center (h,k) and radius r:
$$(x-h)^2 + (y-k)^2 = r^2$$
- Distance from (x_1, y_1) to (x_2, y_2):
$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$
- Area of Triangle: $\frac{1}{2}bh$
- $\sin\theta = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$
- $\cos\theta = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$
- $\tan\theta = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$
- If $\triangle ABC$ is similar to $\triangle DEF$ then
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$
- Volume of Sphere: $\frac{4}{3}\pi r^3$
- Surface Area of Sphere: $4\pi r^2$
- Volume of Cylinder/Prism: (height)(area of base)
- Volume of Cone/Pyramid: $\frac{1}{3}$(height)(area of base)

Theorems

- (IVT) If f is continuous on $[a,b]$, $f(a) \neq f(b)$, and N is between $f(a)$ and $f(b)$ then there exists $c \in (a,b)$ that satisfies $f(c) = N$.

Limits

- $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
- $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$
- $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0$

Derivatives

- $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$
- $(\cot x)' = -\csc^2 x$
- $(\csc x)' = -\csc x \cdot \cot x$

Trigonometric

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
- Table of Trig Values

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(x)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(x)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan(x)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	DNE