Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (8 points) Calculate the following limits.

(a) $\lim_{x\to 1} \frac{x^2-1}{x^2-4x+3}$

Solution:

$$\lim_{x\to 1} \frac{x^2-1}{x^2-4x+3} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-3)(x-1)}$$
$$= \lim_{x\to 1} \frac{(x+1)}{(x-3)} = \frac{2}{-2} = \boxed{-1}$$

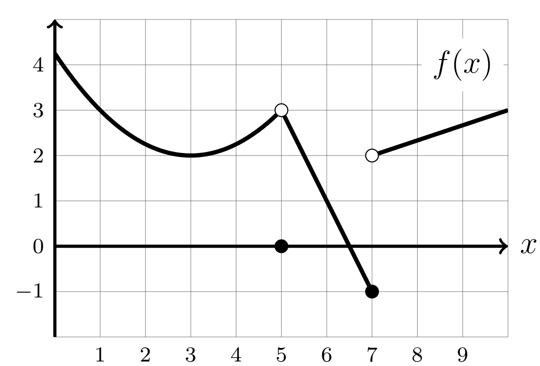
(b) $\lim_{x\to 2} \frac{x+3}{(x-2)^2}$

Solution:

$$\lim_{x\to 2^-} \frac{x+3}{(x-2)^2} = +\infty \qquad \lim_{x\to 2^+} \frac{x+3}{(x-2)^2} = +\infty$$

So therefore $\lim_{x\to 2} \frac{x+3}{(x-2)^2} = \boxed{+\infty}$

2. (6 points) Use the graph of $f(x)$ below to calculate the following limits.



(a) $\lim_{x\to 5} f(x) = \boxed{3}$

(b) $\lim_{x\to 7^-} f(x) = \boxed{-1}$

(c) $\lim_{x\to 7} f(x) = \boxed{\text{DNE}}$

3. (8 points) Calculate the derivative of the following functions.

(a) $f(x) = 4x^3 + \sqrt[3]{x^2} + \dfrac{2}{x^2} + \sec x$

Solution:

$$\boxed{f'(x) = 12x^2 + \tfrac{2}{3}x^{-1/3} - \frac{4}{x^3} + \sec x \tan x}$$

(b) $h(x) = \sin(x^2 + 1)$

Solution:

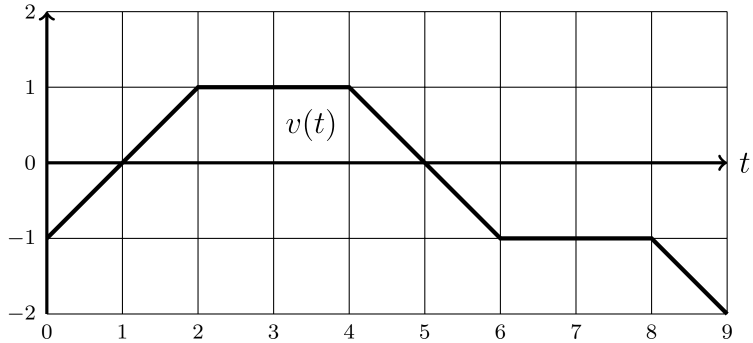
$$\boxed{h'(x) = \cos(x^2 + 1) \cdot (2x)}$$

4. (6 points) $f(x) = \sqrt{x-3}$. Use the **limit definition** of derivative to show that $f'(x) = \dfrac{1}{2\sqrt{x-3}}$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \cdot \left(\frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}} \right) \\
&= \lim_{h\to 0} \frac{h}{h} \cdot \left(\frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} \right) \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} \\
&= \boxed{\frac{1}{2\sqrt{x-3}}}
\end{aligned}$$

5. (10 points) The graph below shows the **velocity** *v(t)* of particle moving in a straight line.



(a) What is the maximum speed? (Recall: speed = |velocity|).

Solution: 2

(b) What is the maximum velocity?

Solution: 1

(c) When is the particle moving forward (i.e. in the positive direction). *Use interval notation.*

Solution: $(1, 5)$

(d) When is the particle speeding up? *Use interval notation.*

Solution: $(1, 2) \cup (5, 6) \cup (8, 9)$

(e) When is the particle moving at a constant velocity? *Use interval notation.*

Solution: $(2, 4) \cup (6, 8)$

6. (4 points) Use the definition of continuity to determine if $f(x)$ is continuous at $x = 1$. State your conclusion and explain your reasoning.

$$f(x) = \begin{cases} x^2 + 2 & x < 1 \\ 4 & x = 1 \\ 2x + 1 & x > 1 \end{cases}$$

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 2 = 3$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x + 1 = 3$$

So $\lim_{x \to 1} f(x) = 3$ and yet $f(1) = 4$ therefore f is $\boxed{\text{not continuous}}$ at $x = 1$.

7. Suppose that y and x satisfy the implicit equation $y^2 = 3xy - x^3$.

   (a) (6 points) Find the the derivative $\frac{dy}{dx}$.

Solution:

$$\begin{aligned} 2y \cdot y' &= 3y + 3x \cdot y' - 3x^2 \\ (2y - 3x)y' &= 3y - 3x^2 \\ \frac{dy}{dx} &= \boxed{\frac{3y - 3x^2}{2y - 3x}} \end{aligned}$$

   (b) (2 points) Find an equation of the tangent line through the point $(-4, 4)$.

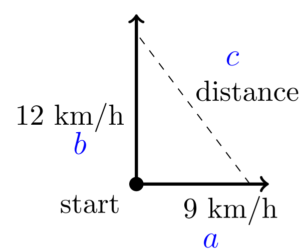
Solution:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-4,4)} = \frac{12 - 3(16)}{8 + 12} = \frac{-36}{20} = \frac{-9}{5}$$

So therefore an equation of the tangent line can be given by

$$\boxed{y - 4 = \tfrac{-9}{5}(x + 4)}$$

8. (6 points) Two boats start sailing from the same point. One boat travels north at 12 km/h and the other travels east at 9 km/h. At what rate is the distance between the boats increasing two hours later?



Solution:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 2aa' + 2bb' &= 2cc' \\ 2(18)(9) + 2(24)(12) &= 2(\sqrt{18^2 + 24^2})c' \\ (18)(9) + (24)(12) &= (\sqrt{18^2 + 24^2})c' \\ c' &= \boxed{\frac{(18)(9) + (24)(12)}{\sqrt{18^2 + 24^2}} \text{ km/hr}} \end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

9. (4 points) Suppose $f(x) = |x + 3|$. For which value of x is f not differentiable?
   - A. $x = 3$
   - B. $x = 1$
   - C. $x = -3$
   - D. $x = 0$
   - E. f is differentiable everywhere.

10. (4 points) Suppose $f(x)$ is a continuous function with values given by the table below.

x	0	1	2	3	4	5
$f(x)$	10.1	3.4	2.9	-1.5	0	0.8

In which interval must there be a c for which $f(c) = 3$?

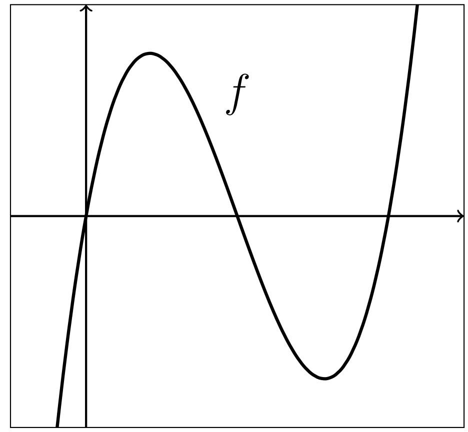
A. $(0, 1)$ B. $(1, 2)$ C. $(2, 3)$ D. $(3, 4)$ E. $(4, 5)$

11. (4 points) If $f(x)$ is a differentiable function, which of the following statements about $f'(1)$ is true?
    - (I) $f'(1)$ is the y-value at $x = 1$.
    - (II) $f'(1)$ is the average rate of change at $x = 1$.
    - (III) $f'(1)$ is the instantaneous rate of change at $x = 1$.
    - (IV) $f'(1)$ is the slope of the secant line at $x = 1$.
    - (V) $f'(1)$ is the slope of the tangent line at $x = 1$.

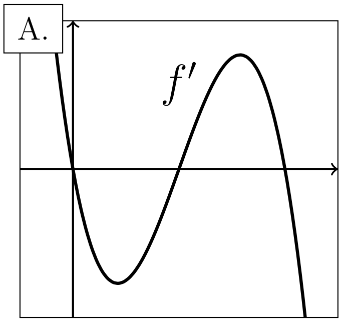
A. (I) only B. (II) only C. (III) only D. (II) and (IV) E. (III) and (V)

12. (4 points) Consider the graph $y = f(x)$ shown to the right.

Which of the following is a graph of its derivative?



A.



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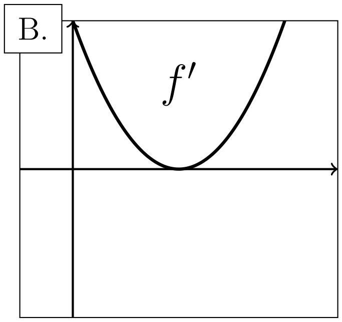
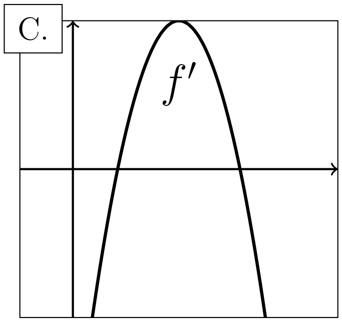
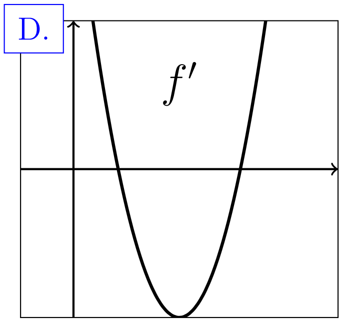


C.



D.



13. (4 points) Let $f(x) = \begin{cases} x^2, & x \leq 2 \\ 2x, & x > 2 \end{cases}$

Which of the following statements is true?

A. The function f is differentiable, but not continuous, at $x = 2$.

B. The function f is continuous and differentiable at $x = 2$.

C. The function f is undefined at $x = 2$.

D. The function f is neither continuous nor differentiable at $x = 2$.

E. The function f is continuous, but not differentiable, at $x = 2$.

14. (4 points) Calculate: $\displaystyle \lim_{x \to 1^-} \frac{x^2 - 1}{|x - 1|}$

A. 2

B. 1

C. $\frac{0}{0}$

D. -2

E. DNE

15. (4 points) Determine the average rate of change of the function $f(t) = 2 + \sin t$ over the interval $\left[0, \frac{\pi}{2}\right]$.

A. $\frac{3\sqrt{3}}{\pi}$

B. $-\frac{1}{\pi}$

C. $\frac{\pi}{12}$

D. $\frac{2}{\pi}$

E. 0

16. (4 points) The velocity $v(t)$ of a particle is shown in the graph below. Which of the following statements is true about the particle motion at $t = 2$?

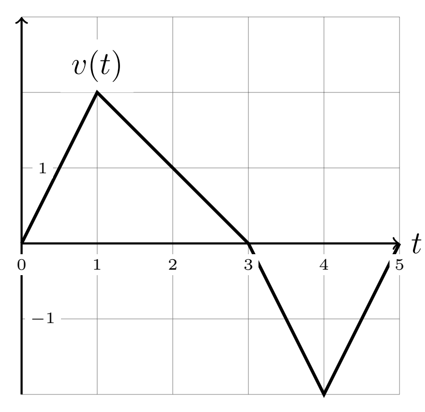
A. The particle is moving backward and speeding up at $t = 2$.

B. The particle is moving backward and slowing down at $t = 2$.

C. The particle is moving forward and speeding up at $t = 2$.

D. The particle is moving forward and slowing down at $t = 2$.

E. None of the above.



17. (4 points) Let $f(1) = 2$, $f'(1) = 3$, $g(1) = 4$, and $g'(1) = 5$. Calculate the following derivative:

$$\frac{d}{dx}\frac{f(x)}{1+g(x)}\bigg|_{x=1}$$

A. $\frac{1}{5}$

B. $-\frac{1}{5}$

C. $-\frac{5}{36}$

D. $\frac{5}{36}$

E. None of the above.

More Challenging Question(s). Show all work to receive credit.

18. (6 points) Calculate: $\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{\sin(4x)}$

Solution:

$$\begin{aligned}\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{\sin(4x)} &= \lim_{x\to 0^+} \frac{\sin\sqrt{x}}{\sin(4x)}\cdot\frac{4x}{4x}\cdot\frac{\sqrt{x}}{\sqrt{x}} \\ &= \lim_{x\to 0^+} \frac{\sin\sqrt{x}}{\sqrt{x}}\cdot\frac{4x}{\sin(4x)}\cdot\frac{\sqrt{x}}{4x} \\ &= (1)\cdot(1)\cdot\lim_{x\to 0^+}\frac{1}{4\sqrt{x}} = \boxed{+\infty}\end{aligned}$$

19. (6 points) Calculate the derivative of $f(x) = \sin^2\left(\frac{\tan x}{x^3}\right)$.

Solution:

$$\boxed{f'(x) = 2\sin\left(\frac{\tan x}{x^3}\right)\cdot\cos\left(\frac{\tan x}{x^3}\right)\cdot\left(\frac{\sec^2(x)\cdot(x^3)-\tan(x)\cdot(3x^2)}{x^6}\right)}$$

20. (2 points) **True** or **False** (*circle one*): $\frac{d}{dx}(\pi^4) = 4\pi^3$

Solution: False