Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

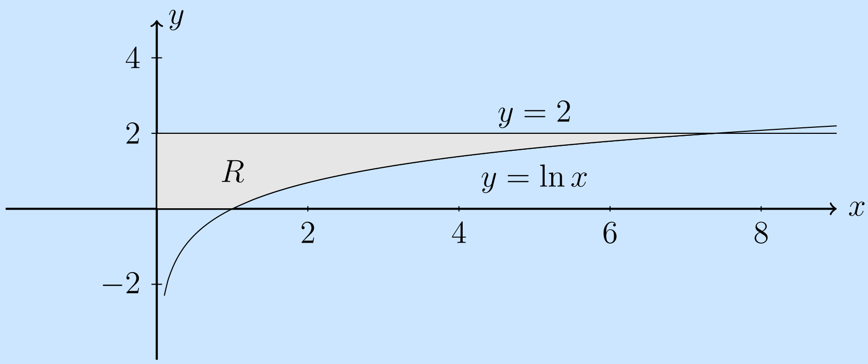
SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Let R be the region in the first quadrant bounded by the curve $y = \ln x$ and $y = 2$.

   (a) (4 points) Sketch and label the region R.

Solution:



   (b) (10 points) Find the volume of the solid formed by revolving the region R around the y-axis.

Solution: Using the expression $V = \int_a^b A(y)\ \mathrm{d}y$, we find $a = 0$ and $b = 2$, with

$$\begin{aligned} A(y) &= \pi x^2 \\ y &= \ln x \\ \implies x &= e^y \\ \implies x^2 &= e^{2y} \\ \implies A(y) &= \pi e^{2y}. \end{aligned}$$

So

$$V = \int_0^2 \pi e^{2y}\ \mathrm{d}y = \frac{\pi}{2} e^{2y} \Big]_0^2 = \frac{\pi}{2}(e^4 - 1).$$

2. (14 points) A tank filled with oil is in the shape of a downward-pointing cone with its vertical axis perpendicular to ground level. Assume that the height of the tank is 15 feet, the circular top of the tank has radius 5 feet and the oil inside the tank weighs 75 pounds per cubic foot. How much work would it take to pump oil from the tank to a level 3 feet above the tank if the tank were completely full?

Solution: Using the formula $W = \int_a^b \rho A(y) D \, dy$. We have that $a = 0$, $b = 15$ are the bounds of integration. The density we read from the question to be $\rho = 75$.

The distance to pump $D = 18 - y$ ft. The cross sectional area is $A(y) = \pi r^2$. Using that the radius is 0 with $y = 0$ and 5 when $y = 15$, for the circular cone we get that the radius $r = y/3$, so the area is $A(y) = \pi(y/3)^2$.

$$\begin{aligned} W &= \int_0^{15} 75\pi \frac{y^2}{9}(18 - y) \, dy \\ &= \int_0^{15} 150\pi y^2 - \frac{25}{3}\pi y^3 \, dy \\ &= 50\pi y^3 - \frac{25}{12}\pi y^4 \Big]_0^{15} \\ &= 50 \cdot 15^3 \pi - \frac{25}{12} \cdot 15^4 \pi \text{ ft-lb.} \end{aligned}$$

3. Evaluate the following integrals.

(a) (7 points) $\displaystyle\int \frac{3x}{(x-2)(x+4)}\,\mathrm{d}x$

Solution: Partial fraction decomposition: $\dfrac{3x}{(x-2)(x+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+4}$.

$\implies 3x = A(x+4) + B(x-2) = (A+B)x + (4A-2B)$

$$\implies \begin{cases} A+B=3 \\ 4A-2B=0 \end{cases} \implies \begin{cases} A=1 \\ B=2 \end{cases}$$

So $\displaystyle\int \frac{3x}{(x-2)(x+4)}\,\mathrm{d}x = \int \frac{1}{x-2} + \frac{2}{x+4}\,\mathrm{d}x = \ln|x-2| + 2\ln|x+4| + C.$

(b) (7 points) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$

Solution: Trig substitution $x = \sec\theta$.
So $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$, and we can simplify $\sqrt{x^2-1} = \tan\theta$.

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x &= \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int \tan^2\theta\,\mathrm{d}\theta = \int \sec^2\theta - 1\,\mathrm{d}\theta \\ &= \tan\theta - \theta + C = \sqrt{x^2-1} - \sec^{-1}x + C\end{aligned}$$

4. (14 points) Evaluate the following limits. (If you use L'Hopital's Rule, explicitly state your reasoning.)

(a) $\lim\limits_{x\to 0} \dfrac{6x^2}{\ln(\sec x)}$

Solution: Directly evaluating the limits of the numerator and denominator gives $\dfrac{6\cdot 0^2}{\ln(\sec(0))} = \dfrac{0}{\ln(1)} = \dfrac{0}{0}$ is indeterminate.
Apply L'Hopital

$$\lim_{x\to 0}\frac{6x^2}{\ln(\sec x)} = \lim_{x\to 0}\frac{12x}{\frac{1}{\sec x}\sec x\tan x} = \lim_{x\to 0}\frac{12x}{\tan x}\left[=\frac{0}{0}\right]$$

another indeterminate form. So apply L'Hopital again

$$=\lim_{x\to 0}\frac{12}{\sec^2 x} = \frac{12}{1} = 12.$$

(b) $\lim\limits_{x\to 0^+} x^{2x}$

Solution: Taking the limits of the base and the exponent gives 0^0 indeterminate form. Since $x^{2x} = e^{2x\ln x}$ evaluate $\lim\limits_{x\to 0^+} 2x\ln x$ (which is an $0\cdot\infty$ indeterminate form) by L'Hopital:

$$\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+}\frac{2\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{2/x}{-x^{-2}} = \lim_{x\to 0^+} -2x = 0.$$

So $\lim\limits_{x\to 0^+} e^{2x\ln x} = e^{\lim_{x\to 0^+} 2x\ln x} = e^0 = 1.$

5. (8 points) Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y\sqrt{x^2+1}}$ with initial value $y(0) = -2$.

Solution: This is a separable differential equation, so we can write $\int y \, \mathrm{d}y = \int \frac{2x}{\sqrt{x^2+1}} \, \mathrm{d}x$. On the left we have $\int y \, \mathrm{d}y = \frac{1}{2}y^2 + C$ and on the right we use u-substitution with $u = x^2 + 1$ and $\mathrm{d}u = 2x \, \mathrm{d}x$ to get

$$\int \frac{2x}{\sqrt{x^2+1}} \, \mathrm{d}x = \int \frac{\mathrm{d}u}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x^2+1} + C.$$

Combining we get $\frac{1}{2}y^2 = 2\sqrt{x^2+1} + C$. Plugging in $y(0) = -2$ yields $\frac{1}{2}(-2)^2 = 2\sqrt{0^2+1} + C$ so $C = 0$. This means $y^2 = 4\sqrt{x^2+1}$. Taking square roots we have $y = \pm\sqrt{4\sqrt{x^2+1}}$; we must take the *negative* of the $\pm$ to satisfy $y(0) = -2$. This leads to the final answer

$$y = -\sqrt{4\sqrt{x^2+1}} \text{ or, equivalently, } -2\sqrt[4]{x^2+1}.$$

6. (6 points) Let $f(x) = 3^{\tan^{-1}(x)}$. Using that $f(1) = 3^{\pi/4}$, compute $(f^{-1})'(3^{\pi/4})$.

Solution: Using the formula $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$, and that $f^{-1}(3^{\pi/4}) = 1$ is given, we compute $f'(1)$. Writing $f(x) = e^{\tan^{-1}(x)\cdot\ln(3)}$ we have

$$f'(x) = \ln(3)3^{\tan^{-1}(x)}(\tan^{-1}(x))' = \ln(3)3^{\tan^{-1}(x)}\frac{1}{1+x^2}$$
$$f'(1) = \ln(3)3^{\tan^{-1}(1)}\frac{1}{1+1^2} = \frac{1}{2}\ln(3) \cdot 3^{\pi/4}$$
$$(f^{-1})'(3^{\pi/4}) = \frac{1}{f'(1)} = \frac{2}{\ln 3 \cdot 3^{\pi/4}}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) Shown is the graph of a force function (in Newtons) acting on a particle as it travels from $x = 0$ m to $x = 10$ m. How much work is done by the force?

A. 200 J

B. 230 J

C. 310 J

D. 400 J

E. 560 J

8. (4 points) If $f(x) = \log_3 x$, find $f'(e)$:

A. $\frac{1}{3}$

B. $\frac{1}{e \ln 3}$

C. 1

D. $\frac{1}{e}$

E. $\frac{1}{3 \log_3 e}$

9. (4 points) Evaluate the integral $\int_1^e t \ \ln t \ dt$.

A. $\frac{1}{2}e^2 + \frac{1}{3}$

B. $\frac{2}{7}e^2$

C. $e - \frac{3}{5}$

D. $\frac{3}{4}e^2 + \frac{1}{4}$

E. $\frac{1}{4}e^2 + \frac{1}{4}$

10. (4 points) If $f(x) = (\sqrt{x})^{\sqrt{x}}$, find $f'(4)$.

A. 2

B. $\ln 2 + 1$

C. 1

D. $(\ln 2 + 2)/4$

E. $4(\ln 2 + 1)$

11. (4 points) Find $y(2)$ if $y''(x) = \sinh x$, $y'(0) = 0$, and $y(0) = 0$.

A. 0

B. $\sinh 2$

C. $\dfrac{\sinh^3 2}{6}$

D. $\sinh 2 - 2$

E. $-\sinh 2 + 2$

12. (4 points) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x \; dx$.

A. $-1/24$

B. $-1/6$

C. 0

D. $1/6$

E. $1/24$

13. (4 points) Evaluate the integral $\int_1^\infty \frac{1}{x^2} \sin\left(\frac{1}{x}\right) \mathrm{d}x$.

A. 0

B. $\sin 1$

C. $1 - \cos 1$

D. 1

E. The integral is divergent.

14. (4 points) Evaluate the integral $\int_{-e}^{e} \frac{1}{|x|} \mathrm{d}x$.

A. 0

B. $2 \ln 2$

C. 2

D. $2e$

E. The integral is divergent.

15. (4 points) A roast turkey is removed from the oven when its temperature reached 195°F. It is left on the counter where the air temperature is 75°F. After 1 hour, the temperature of the turkey dropped to 135°F. What is the temperature of the turkey after *another* 1 hour? (Remember: Newton's Law of Cooling is $T'(t) = k(T(t) - T_s)$).

A. 75°F

B. 105°F

C. $135^2/195$ °F

D. $75^2/135$ °F

E. $75 \times 195/135$ °F

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	14	
7	12	
8	12	
9	12	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$
- $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\ dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

Derivatives

- $\dfrac{d}{dx}(\sinh x) = \cosh x \qquad \dfrac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\cos^2 x + \sin^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$