Name: _____

Section: _____ **Recitation Instructor**: _____

INSTRUCTIONS

- Fill in your name, etc. on this first page.

- Without fully opening the exam, check that you have pages 1 through 11.

- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.

- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.

- If you have any questions please raise your hand.

- You will be given exactly 90 minutes for this exam.

- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.

- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).

- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.

- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.

- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the
above instructions and statements
regarding academic honesty: _____
SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Evaluate the following integrals. Show all work.

(a) (7 points) $\displaystyle\int \frac{1}{1+9x^2}\, dx$

Solution: Use trig substitution with

$$3x = \tan(\theta), \qquad 3\, dx = \sec^2(\theta)\, d\theta, \qquad 1+9x^2 = 1+(3x)^2 = \sec^2(\theta).$$

Integral becomes

$$\implies \int \frac{1}{\sec^2(\theta)}\cdot\frac{1}{3}\sec^2(\theta)\, d\theta = \int \frac{1}{3}\, d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\tan^{-1}(3x)$$

Alternatively: use u substitution with $u = 3x$ and use the formula sheet which shows $\displaystyle\int \frac{1}{1+u^2}\, du = \tan^{-1}(u) + C$.

(b) (7 points) $\displaystyle\int x\cosh(x)\, dx.$

Solution: Use integration by parts, with $u = x$ and $dv = \cosh(x)\, dx$ to get

$$\begin{aligned} u &= x & du &= dx \\ v &= \sinh(x) & dv &= \cosh(x)\, dx \end{aligned}$$

and

$$\int x\cosh(x)dx = uv - \int v\, du = x\sinh(x) - \int \sinh(x)\, dx = x\sinh(x) - \cosh(x) + C.$$

2. Evaluate the following limits. (If you use L'Hopital's Rule, explicitly state your reasoning.)

(a) (7 points) $\lim\limits_{x \to \infty} \ln\left(x^{1/\sqrt{x}}\right)$

Solution:

$$
\begin{aligned}
\lim_{x \to \infty} \ln\left(x^{1/\sqrt{x}}\right) &= \lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} &&\text{First simplify} \\[2mm]
&\left(= \frac{\infty}{\infty}\right) &&\text{Check is indeterminate form} \\[2mm]
&\overset{\text{L'H}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} &&\text{L'Hopital's rule} \\[2mm]
&= \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} &&\text{Simplify} \\[2mm]
&= 0
\end{aligned}
$$

(b) (7 points) $\lim\limits_{y \to 0} \dfrac{1 - \cos(4y)}{e^{8y} - 1}$

Solution:

$$
\begin{aligned}
&\left(= \frac{1-1}{1-1} = \frac{0}{0}\right) &&\text{Check is indeterminate form} \\[2mm]
\lim_{y \to 0} \frac{1 - \cos(4y)}{e^{8y} - 1} &\overset{\text{L'H}}{=} \lim_{y \to 0} \frac{4\sin(4y)}{8e^{8y}} &&\text{L'Hopital's rule} \\[2mm]
&= \frac{4 \cdot 0}{8 \cdot 1} = 0 &&\text{Compute the limit}
\end{aligned}
$$

3. (7 points) A 500 lb anchor hangs off of an airship from a chain that is 100 ft long. The chain itself weighs another 200 lbs (in addition to the weight of the anchor). How much work will it take to pull the anchor up onto the deck of the airship by the chain?

Solution:

1. Density of the chain to is $\dfrac{200\text{lbs}}{100\text{ft}} = 2$ lbs/ft

2. Work done to pull up anchor itself: 500 lbs $\times$ 100 ft $= 50{,}000$ ft-lbs.
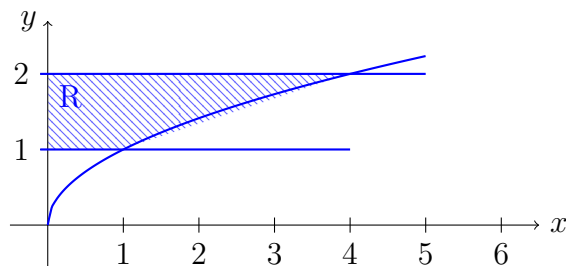
3. Work done to pull up chain:

$$\int_0^{100} 2(100 - y)\ dy = 200y - y^2 \Big|_0^{100} = 20000 - 10000 = 10000.$$

4. Total work $= 60{,}000$ ft-lbs.

4. Let R be the region bounded by the curves $y = \sqrt{x}$, $y = 1$, $y = 2$, and the y-axis.

    (a) (2 points) Sketch the region R; make sure to shade and label the region R.

    (b) (5 points) Find the volume of the solid formed by revolving R around the y-axis.

Solution:

1. The area of a thin slice of the solid of rotation at height y is $\pi x^2 = \pi(y^2)^2$.

2. The volume of the thin slice is $\pi y^4\ dy$.

3. The total volume is therefore $= \displaystyle\int_1^2 \pi y^4\ dy$.

4. Final answer is $31\pi/5$

5. (7 points) Give the partial fraction decomposition of $\dfrac{x}{x^2 + 3x + 2}$.

Solution:

1. Factor the denominator
$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x+1)(x+2)}$$

2. Setup for partial fraction decompostion
$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

3. Cross multiply gets
$$A(x+2) + B(x+1) = (A+B)x + (2A+B)$$
this needs to be equal to the numerator x.

4. So solve
$$A + B = 1 \qquad 2A + B = 0$$
and get
$$A = -1 \text{ and } B = 2$$

5. So that
$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

6. (7 points) Evaluate the integral $\displaystyle\int_2^\infty \frac{1}{x-1} - \frac{1}{x+1}\ dx$.

Solution:

1. Treat improper integral with limits
$$\int_2^\infty \frac{1}{x-1} - \frac{1}{x+1}\ dx = \lim_{a\to\infty} \int_2^a \frac{1}{x-1} - \frac{1}{x+1}\ dx$$

2. Compute the integral
$$\lim_{a\to\infty} \int_2^a \frac{1}{x-1} - \frac{1}{x+1}\ dx = \lim_{a\to\infty} \left[\ln(a-1) - \ln(a+1) - \ln(1) + \ln(3)\right]$$

3. Simplify the limit using rules of logarithm
$$\lim_{a\to\infty} \left[\ln(a-1) - \ln(a+1) - 0 + \ln(3)\right] = \ln(3) + \ln\left(\lim_{a\to\infty} \frac{a-1}{a+1}\right)$$

4. Compute the limit
$$= \ln(3) + \ln(1) = \ln(3)$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) Evaluate the integral $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$.

A. $\dfrac{\sqrt{1-x^2}}{x} + C$

B. $-\dfrac{\sqrt{1-x^2}}{x} + C$

C. $\dfrac{\sqrt{1-x^2}}{x^2} + C$

D. $-\dfrac{\sqrt{1-x^2}}{x^2} + C$

E. $-x\sqrt{1-x^2} + C$

8. (4 points) $\sin\left(\tan^{-1}\left(\dfrac{\sqrt{5}}{2}\right)\right) = ?$

A. $\dfrac{2}{\sqrt{5}}$

B. $\dfrac{2}{3}$

C. $\dfrac{\sqrt{5}}{3}$

D. $\dfrac{3}{2}$

E. $\dfrac{3}{\sqrt{5}}$

9. (4 points) The derivative of $f(x) = x^{\cos(x)}$ is

A. $\sin(x)\ln x + \dfrac{\cos(x)}{x}$

B. $-\sin(x)\ln x + \dfrac{\cos(x)}{x}$

C. $\left(\sin(x)\ln x + \dfrac{\cos(x)}{x}\right) x^{\cos(x)}$

D. $\left(-\sin(x)\ln x + \dfrac{\cos(x)}{x}\right) x^{\cos(x)}$

E. $\left(-\sin(x)\ln x + \dfrac{\cos(x)}{x}\right) x^{\cos(x)} + C$

10. (4 points) Solve the initial value problem $\dfrac{dy}{dx} = 1 - 2y$ with initial value $y(0) = 0$.

     A. $y = \ln(1 - 2x)$.

     B. $y = -\dfrac{1}{2}\ln(1 - 2x)$.

     C. $y = 1 - e^{-x}$.

     D. $y = \dfrac{e^{-\frac{1}{2}x} - 1}{2}$.

     E. $y = \dfrac{1 - e^{-2x}}{2}$.

11. (4 points) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x \cos^3 x \; dx$.

     A. $1/40$

     B. $1/24$

     C. 0

     D. $-1/24$

     E. $-1/40$

12. (4 points) Compute $f'(x)$ if $f(x) = 2^{\tan^{-1}(x)}$.

     A. $2^{\tan^{-1}(x)} \cdot \ln(2) \cdot \sec^2(x)$

     B. $2^{\tan^{-1}(x)} \cdot \dfrac{\ln 2}{1 + x^2}$

     C. $2^{\tan^{-1}(x)-1} \cdot \dfrac{\ln 2}{1 + x^2}$

     D. $2^{\tan^{-1}(x)} \cdot \dfrac{1}{1 + x^2}$

     E. $2^{\tan^{-1}(x)-1} \cdot \dfrac{1}{1 + x^2}$

13. (4 points) 5 days ago, a sample of radioactive substance is measured at 200 grams. Today, the sample measures at 100 grams. How many days from now will there be only 25 grams of the sample remaining?

     A. 3.75

     B. 5

     C. 7.5

     D. 10

     E. 15

14. (4 points) Let $f(x) = \dfrac{e^{-x} + 2}{x}$. Which of the following statements is correct concerning the improper integral $\displaystyle\int_1^\infty f(x)\,dx$?

     A. Since $f(x) \geq \dfrac{1}{x^2}$, by the comparison test the integral converges.

     B. Since $f(x) \geq \dfrac{1}{x}$, by the comparison test the integral diverges.

     C. Since $f(x) \leq \dfrac{3}{x}$, by the comparison test the integral converges.

     D. Since $f(x) \leq 2 + e^{-x}$, by the comparison test the integral converges.

     E. The comparison test cannot be used.

15. (4 points) If $f(x) = 4x + \cos(x)$, find $(f^{-1})'(1)$, knowing that $f(0) = 1$.

     A. $\dfrac{1}{2}$.

     B. $\dfrac{1}{4}$.

     C. $\dfrac{1}{3}$.

     D. $-\dfrac{1}{4}$.

     E. $-\dfrac{1}{2}$.

More Challenging Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

16. (7 points) You and your friend both evaluate an integral and get different answers. You get that the integral is $\tan^2(x)/2 + C$, and your friend gets that the integral is $\sec^2(x)/2 + C$. Your TA tells the two of you that you are both correct. Use the constant of integration to explain how this can be the case even though $\tan(x) \neq \sec(x)$.

> **Solution:**
> The two functions $\dfrac{1}{2}\sec^2(x)$ and $\dfrac{1}{2}\tan^2(x)$ differs by a constant (in fact $\dfrac{1}{2}$) due to the trig identity
>
> $$\sec^2(x) = \tan^2(x) + 1$$
>
> Since the constant of integration is arbitrary, both answers are correct.

17. (7 points) Explain why $\displaystyle\int_{-100}^{100} \sinh(\sin(x))\ dx = 0$. *Hint: Don't try to evaluate this integral. Instead, start by thinking about how $\sin(-x)$ relates to $\sin(x)$.*

> **Solution:** the function $\sin(x)$ is an odd function.
>
> the function $\sinh(x)$ is an odd function.
>
> the composition of odd functions is still odd: so $\sinh(\sin(x))$ is an odd function.
>
> integrating any odd function on an interval $(-a, a)$ is always zero.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x) \; dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x) \; dx$$

- $\displaystyle\int \frac{1}{x} \; dx = \ln|x| + C$

- $\displaystyle\int \tan x \; dx = \ln|\sec x| + C$

- $\displaystyle\int \sec x \; dx = \ln|\sec x + \tan x| + C$

- $\displaystyle\int a^x \; dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$

- **Integration by Parts**:

$$\int u \; dv = uv - \int v \; du$$

Derivatives

- $\dfrac{d}{dx}(\sinh x) = \cosh x$ $\dfrac{d}{dx}(\cosh x) = \sinh x$

- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$

- $\cos^2 x + \sin^2 x = 1$

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

- $\sin(2x) = 2\sin x \cos x$

- $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$