1. Determine if the following series converge or diverge. **If the series converges, also compute the sum**. You must show all of your work and support your conclusions.

(a) (7 points) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+1}}$

Solution:

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{6} \cdot \left(\frac{2}{3}\right)^n$$

So by the Geometric Series Test and since $r = 2/3$ this converges. And it converges to:

$$= \frac{1/6}{1 - 2/3}$$
$$= \boxed{\frac{1}{2}}$$

(b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution:
Using the integral test we note that

1. Positive: Since $n \geq 2$ we know $n > 0$ and $\sqrt{\ln n} > 0$ so $\dfrac{1}{n\sqrt{\ln n}} > 0$.
2. Continuous: Since $n \geq 2$ and notably perhaps $n \neq 0$ and $n > 1$ we know that $\dfrac{1}{n\sqrt{\ln n}}$ is continuous on this domain
3. Decreasing: Since $n\sqrt{\ln n}$ is an increasing function, $\dfrac{1}{n\sqrt{\ln n}}$ is decreasing

So the hypotheses of the integral test are satisfied. Moreover:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx &= \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\, du \\ &= \lim_{t\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{t} \\ &= \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{\ln 2} \to \infty \qquad \text{(DIVERGES)} \end{aligned}$$

and so since the improper integral diverges our series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ also $\boxed{\text{diverges}}$.

2. Determine if the following series converge or diverge. You must show all of your work and justify your use of any series convergence tests.

   (a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

   Solution:

   Consider the ratio test

   $$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} \right| = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

   Since this limit $L < 1$ by the ratio test the series must $\boxed{\text{converge}}$.

   (b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^2+n^3}$

   Solution:

   Consider the LCT with $b_n = \dfrac{\sqrt{n^4}}{n^3} = \dfrac{n^2}{n^3} = \dfrac{1}{n}$

   $$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2+n^3} \cdot \frac{n^3}{\sqrt{n^4}} &= \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{\sqrt{n^4}} \cdot \frac{n^3}{n^2+n^3} \\ &= \lim_{n\to\infty} \sqrt{\frac{n^4+1}{n^4}} \cdot \frac{n^3}{n^3+n^2} = \sqrt{1} \cdot 1 = 1 \end{aligned}$$

   By the LCT since $\sum b_n = \sum \dfrac{1}{n}$ diverges by the p-series test, our series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^2+n^3}$ must also $\boxed{\text{diverge}}$.

3. (7 points) Find the open interval of convergence for the power series $\displaystyle\sum_{n=5}^{\infty} \frac{(2x+3)^n}{n^2-2}$.

(*You do not have to test the end points for convergence.*)

Solution: Use the ratio test

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{(n+1)^2-2} \cdot \frac{n^2-2}{(2x+3)^n} \right| &= \lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{(2x+3)^n} \cdot \frac{n^2-2}{(n+1)^2-2} \right| \\
&= \lim_{n\to\infty} |2x+3| \cdot \frac{n^2-2}{n^2+2n-1} \\
&= |2x+3| \cdot (1)
\end{aligned}$$

By the ratio test this will converge if $|2x+3| < 1$. So

$$\begin{aligned}
-1 < 2x+3 < 1 \\
-4 < 2x < -2 \\
-2 < x < -1
\end{aligned}$$

Giving us the open interval of convergence $\boxed{(-2,-1)}$.

4. (7 points) Find the Taylor polynomial of degree 3 for $f(x) = \cos(2\sqrt{x})$ centered at $a = 0$.

Solution: Using the Maclaurin expansion for $\cos u$ we have

$$\begin{aligned}
\cos u &\approx 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} \\
\cos(2\sqrt{x}) &\approx 1 - \frac{(2\sqrt{x})^2}{2!} + \frac{(2\sqrt{x})^4}{4!} - \frac{(2\sqrt{x})^6}{6!} \\
&\approx \boxed{1 - \frac{2^2}{2!}x + \frac{2^4}{4!}x^2 - \frac{2^6}{6!}x^3} \\
&\approx \boxed{1 - 2x + \frac{2}{3}x^2 - \frac{4}{45}x^3}
\end{aligned}$$

5. (7 points) Find the Maclaurin series (Taylor series centered at $a = 0$) representation of $f(x) = \dfrac{x}{1+2x^4}$.
Express your answer in sigma notation.

Solution:
Let $u = -2x^4$ then

$$\frac{x}{1+2x^4} = x\left(\frac{1}{1-u}\right) = x\left(\sum_{n=0}^{\infty} u^n\right) = x\left(\sum_{n=0}^{\infty}(-2x^4)^n\right)$$
$$= x\left(\sum_{n=0}^{\infty}(-2)^n x^{4n}\right)$$
$$= \boxed{\sum_{n=0}^{\infty}(-2)^n \cdot x^{4n+1}}$$

6. (7 points) Evaluate the indefinite integral $\displaystyle\int \frac{\tan^{-1} x}{x}\, dx$ as a power series.
Express your answer in sigma notation and state the radius of convergence of the series.

Solution: If you have memorized the power series expansion for $\tan^{-1}(x)$ from the notes/book you may use it, otherwise you need to integrate

$$\int \frac{dx}{1+x^2} = \int\left(\sum_{n=0}^{\infty}(-x^2)^n\right)\, dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)\, dx$$

to find the series expansion for $\tan^{-1}(x)$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot x^{2n+1}$$
$$\frac{\tan^{-1}(x)}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot x^{2n}$$
$$\int\frac{\tan^{-1}(x)}{x}\, dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot x^{2n}\right)\, dx$$
$$= \boxed{\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cdot x^{2n+1}\right] + C}$$

Since the radius of convergence is $\boxed{R = 1}$ (same as for $\dfrac{1}{1-u}$)

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}(-1)^n}{\ln(n)+1}$?

   A. The **alternating series test** shows that the series converges.

   B. The **$\mathbf{n^{th}}$-term test** shows that the series diverges.

   C. The **ratio test** shows that the series converges.

   D. The **integral test** shows that the series diverges.

   E. None of the above are true.

8. (4 points) Find the limit of the **sequence** a_n where the n^{th} term is given by $a_n = \frac{\tan^{-1} n}{\sqrt[2n]{n}}$.

   A. $\pi/2$

   B. $\pi/4$

   C. 1

   D. 0

   E. The sequence diverges

9. (4 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1) \sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}+1} \quad \text{and} \quad (2) \sum_{n=2}^{\infty} \frac{(-1)^n}{2\ln n}$$

   A. (1) is absolutely convergent; (2) is divergent.

   B. (1) is conditionally convergent; (2) is divergent.

   C. (1) is absolutely convergent; (2) is conditionally convergent.

   D. (1) is divergent; (2) is conditionally convergent.

   E. Both (1) and (2) are conditionally convergent.

10. (4 points) The Taylor series of the function $f(x)$, centered at $a = 1$, is given by $\sum_{n=0}^{\infty} \frac{n^2-1}{n!}(x-1)^n$. What is the value of the third derivative $f'''(1)$?

A. 0

B. 4/3

C. 3/2

D. 8

E. 3

11. (4 points) The first 3 nonzero terms of the Taylor series, centered at $a = 0$, for $f(x) = \frac{x^3}{1-x^2}$ are

A. $x^3 - x^5 + x^7$

B. $x^3 + x^5 + x^7$

C. $\frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

D. $1 + x^2 + x^3$

E. $\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$

12. (4 points) For which values of x does the series $\sum_{n=0}^{\infty} e^{nx}$ converge ?

A. $x < 1$

B. $-1 < x < 1$

C. $0 < x < 1$

D. $-1 < x < 0$

E. $x < 0$

13. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{\sqrt{\ln(n)}}{n^{3/2}}$?

A. By the ratio test, the series converges.

B. By the ratio test, the series diverges.

C. The ratio test is inconclusive for this series, but the series diverges by another test.

D. The ratio test is inconclusive for this series, but the series converges by another test.

E. None of the above are true.

14. (4 points) What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$?

A. 0

B. 5

C. 2

D. 5/2

E. ∞

15. (4 points) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$ and diverges at $x = -2$. Which of the following statements must also be true?

A. The power series also converges at $x = 5$.

B. The power series also converges at $x = 0$.

C. The power series also diverges at $x = 7$.

D. The power series converges at $x = 6$ only.

E. None of the above.

More Challenging Questions. Choose only the correct answer.

16. (4 points) Which statement about the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ is true?

A. It converges by the integral test.

B. It diverges by a comparison to the divergent series $\sum_{n=2}^{\infty} \frac{1}{n}$.

C. It converges by a comparison to the convergent series $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

D. It diverges by the p-series test with $p = 1$.

E. None of the above.

Answer the True/False Questions

17. (4 points) Suppose $\{a_n\}_{n=0}^{\infty}$ is a convergent sequence with limit A.

- The series $\sum (-1)^n a_n$ always converges. **T** **F**

Solution: FALSE. One such case is $a_n = 1$ with $A = 1$

- The series $\sum (a_n - A)$ always converges. **T** **F**

Solution: FALSE. One such case is $a_n = 1/n$ with $A = 0$

18. (3 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ **T** **F**

Solution: TRUE. From power series expansion of e^x with $x = -1$

19. (3 points) $\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{24}$ **T** **F**

Solution: FALSE. From power series expansion of $\sin(x)$ the limit should be $1/5! = 1/120$.
Also could use L'Hopitals rule 5 times to get the same result.