Name: ____________________

Section: ____________ **Recitation Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) King Kong is fishing for airplanes atop the Empire State Building using an elevator cable that weighs 2 lb/ft. He catches one that weighs 4000 lbs. How much work does it take King Kong to reel in the airplane, raising it 800 ft in the process?

Solution: We can set up a coordinate axis with the origin at the tip of Kong's fishing pole. We then note that

1. Each dy ft section of the cable weights 2 dy lbs
2. Each section of cable at y ft below the tip of the pole needs to be lifted y ft.
3. The plane itself will need $4000 * 800 = 3,200,000$ ft-lbs to be lifted to the tip of the pole.

So, the total work is $3,200,000 + \int_0^{800} 2y\ dy$ ft-lbs $= 3,840,000$ ft-lbs

2. Let R be the region bounded by the curves $y = \ln(x/2)$, $x = 0$, $y = -1$, and $y = 1$.

(a) (3 points) Sketch the region R; make sure to label the x-intercept(s) and shade the region R.

Solution: Check for:

1. the x-intercept at $x = 2$,
2. for the basic shape of $\ln(x)$ as a concave function, and
3. for the properly shaded region.

(b) (4 points) Set-up, but do not evaluate, an integral representing the volume of the solid formed by revolving R around the y-axis.

Solution: At position y along the y-axis we integrate from $x = 0$ to $x = 2e^y$. Thus, the integral is

$$\int_{-1}^{1} \pi(2e^y)^2\ dy$$

3. (7 points) Let $f(x) = x^3 + 2x + \frac{1}{\pi}\sin(\pi x)$. Knowing that $f(1) = 3$, what is the value of $(f^{-1})'(3)$?

Solution: Using the formula sheet we have that

$$\begin{aligned}(f^{-1})'(3) &= \frac{1}{f'(f^{-1}(3))} \\ &= \frac{1}{f'(1)}.\end{aligned}$$

Since $f'(x) = 3x^2 + 2 + \cos(\pi x)$ we can also see that $f'(1) = 5 - 1 = 4$.

As a result we have that $(f^{-1})'(3) = \frac{1}{4}$.

4. (7 points) Compute the derivative of the function $g(x) = \big(2 + \cosh(x)\big)^{3x+3}$.

Solution: We have that $g(x) = e^{\ln(2+\cosh(x))(3x+3)}$ so that

$$\begin{aligned}g'(x) &= e^{\ln(2+\cosh(x))(3x+3)} \cdot (\ln(2+\cosh(x))(3x+3))' \\ &= e^{\ln(2+\cosh(x))(3x+3)} \cdot \left(\frac{3x+3}{2+\cosh(x)} \cdot (\cosh(x))' + 3\ln(2+\cosh(x))\right) \\ &= e^{\ln(2+\cosh(x))(3x+3)} \cdot \left(\frac{3x+3}{2+\cosh(x)} \cdot \sinh(x) + 3\ln(2+\cosh(x))\right) \\ &= \big(2+\cosh(x)\big)^{3x+3} \cdot \left(\frac{3x+3}{2+\cosh(x)} \cdot \sinh(x) + 3\ln(2+\cosh(x))\right)\end{aligned}$$

5. Evaluate the following integrals. Show all work.

(a) (7 points) $\int \sin^3 x \ \cos^4 x \ dx$

Solution:

$$\begin{aligned}\int \sin^3 x \ \cos^4 x \ dx &= \int \sin x(1-\cos^2(x)) \ \cos^4 x \ dx \\ &= \int \sin x \ (\cos^4 x - \cos^6 x) \ dx\end{aligned}$$

Letting $u = \cos(x), \ du = -\sin x \ dx$ and integrating we get

$$\begin{aligned}\int \sin^3 x \ \cos^4 x \ dx &= \int u^6 - u^4 \ du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\ &= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C\end{aligned}$$

(b) (7 points) $\int \frac{1}{x^2+x-2} \ dx.$

Solution: We first factor the denominator to get that $x^2+x-2 = (x+2)(x-1)$. We now use partial fractions to set that

$$\int \frac{1}{x^2+x-2} \ dx = \int \frac{A}{x+2} + \frac{B}{x-1}$$

where $A(x-1)+B(x+2) = 1$. This means that $A+B=0$ and $2B-A=1$ both hold, which implies that $A = -B = \frac{-1}{3}$ and $B = \frac{1}{3}$. As a result,

$$\begin{aligned}\int \frac{1}{x^2+x-2} \ dx &= \int \frac{-1}{3(x+2)} + \frac{1}{3(x-1)} + C \\ &= -\frac{1}{3}\left(\ln(|x+2|) - \ln(|x-1|)\right) + C \\ &= \frac{1}{3}\ln\left(\left|\frac{x-1}{x+2}\right|\right) + C\end{aligned}$$

6. (7 points) Evaluate the integral $\int_0^5 \frac{x}{x-2}\, dx$.

Solution: We need to notice that this is an improper integral.

$$\begin{aligned}
\int_0^5 \frac{x}{x-2}\, dx &= \lim_{a\to 2^-} \int_0^a \frac{x}{x-2}\, dx + \lim_{b\to 2^+} \int_b^5 \frac{x}{x-2}\, dx \\
&= \lim_{a\to 2^-} \int_0^a 1 + \frac{2}{x-2}\, dx + \lim_{b\to 2^+} \int_b^5 1 + \frac{2}{x-2}\, dx \\
&= \lim_{a\to 2^-} [x + 2\ln|x-2|]_0^a + \lim_{b\to 2^+} [x + 2\ln|x-2|]_b^5 \\
&= \lim_{a\to 2^-} a + 2\ln|a-2| - 2\ln 2 + \lim_{b\to 2^+} 5 + 2\ln 3 - b - 2\ln|b-2|
\end{aligned}$$

This **DNE**, however, since $\lim_{a\to 2^-} \ln|a-2| = -\infty$ and $\lim_{b\to 2^+} \ln|b-2| = -\infty$.

7. (7 points) Solve the initial value problem $\frac{dy}{dx} = \sqrt{1-9y^2}$ with initial value $y(-1) = 0$.

Solution: The equation is separable. We have that

$$\begin{aligned}
\int \frac{dy}{\sqrt{1-9y^2}} &= \int\, dx \\
\int \frac{dy}{\sqrt{1-(3y)^2}} &= \int\, dx \\
\frac{1}{3}\sin^{-1}(3y) &= x + C \\
3y &= \sin(3x + \tilde{C}) \\
y &= \frac{1}{3}\sin(3x + \tilde{C})
\end{aligned}$$

Taking the initial value into account we see that

$$0 = \sin(-3 + \tilde{C}) \implies -3 + \tilde{C} = n\pi \implies \tilde{C} = 3 + n\pi$$

where n is any integer. Thus,

$$y(x) = \frac{1}{3}\sin(3x + 3 + n\pi)$$

for any integer n.

Note: We only expect them to report one value of n that works, e.g., $n = 0$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Evaluate the limit $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\ln x}$.

A. 0.

B. 1.

C. -1.

D. The limit does not exist: it tends to ∞.

E. The limit does not exist: it tends to $-\infty$.

9. (4 points) Evaluate the limit $\lim_{x\to\infty} x\left(\tan^{-1}(x) - \frac{\pi}{2}\right)$.

A. 0.

B. 1.

C. -1.

D. The limit does not exist: it tends to ∞.

E. The limit does not exist: it tends to $-\infty$.

10. (4 points) Which statement is **FALSE**?

A. $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

B. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ diverges.

C. $\int_1^\infty \frac{1}{x}\,dx$ diverges.

D. $\int_0^1 \ln x\,dx$ converges.

E. $\int_0^3 \frac{1}{x-1}\,dx$ diverges.

11. (4 points) The population in a bacterial culture grows exponentially. A culture started with 1000 cells. After 5 hours 3000 cells were observed. After how many *additional* hours will there be 6000 cells?

A. $\frac{15}{2}$.

B. 5.

C. $\frac{5\ln 3}{\ln 2}$.

D. $\frac{10}{3}$.

E. $\frac{5\ln 2}{\ln 3}$.

12. (4 points) $\cos\left(\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) = ?$

A. $\frac{2}{\sqrt{5}}$

B. $\frac{2}{3}$

C. $\frac{\sqrt{5}}{3}$

D. $\frac{3}{2}$

E. $\frac{3}{\sqrt{5}}$

13. (4 points) Choose the correct partial fraction decomposition of $\frac{x^3}{x^2-4}$.

A. $x + \frac{2}{x+2} + \frac{2}{x-2}$

B. $x + \frac{2}{x+2} - \frac{2}{x-2}$

C. $x - \frac{2}{x+2} + \frac{2}{x-2}$

D. $x - \frac{2}{x+2} - \frac{2}{x-2}$

E. $-x + \frac{2}{x+2} - \frac{2}{x-2}$

14. (4 points) Evaluate the definite integral $\frac{1}{4}\int_0^{\pi/2} \sin^2(2x)\, dx$.

A. 0

B. 1

C. $\dfrac{\pi}{16}$

D. $\dfrac{\pi}{8} - \dfrac{3\pi}{4}$

E. $\dfrac{\pi - 2}{2\pi^2}$

15. (4 points) Evaluate the definite integral $\int_0^{1/2} x\cos(\pi x)\, dx$.

A. 0

B. 1

C. $\dfrac{\pi}{16}$

D. $\dfrac{\pi}{8} - \dfrac{3\pi}{4}$

E. $\dfrac{\pi - 2}{2\pi^2}$

16. (4 points) Let $f(x) = \dfrac{\cos(x) + 1}{(x + \sqrt{x})^2}$. Which of the following statements is correct concerning the improper integral $\int_2^{\infty} f(x)\, dx$?

A. Since $f(x) \geq \dfrac{1}{x^4}$, by the comparison test the integral diverges.

B. Since $f(x) \leq \dfrac{2}{x^2}$, by the comparison test the integral converges.

C. Since $f(x) \leq \dfrac{2}{x}$, by the comparison test the integral converges.

D. Since $f(x) \geq \dfrac{1}{x}$, by the comparison test the integral diverges.

E. The comparison test cannot be used because $f(x)$ changes signs.

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

17. (7 points) Suppose that a function f has $0 \leq f(x) \leq \dfrac{1}{x^3}$ for all $x > 0$, explain why the integral $0 \leq \displaystyle\int_3^\infty f(x)\,dx \leq \frac{1}{18}$.

Solution: We want to use the *Comparison Theorem for Integrals* from Section 7.8. Appealing to that along with our knowledge that $0 \leq f(x) \leq \dfrac{1}{x^3}$ for all $x > 0$ we have

$$\begin{aligned} \int_3^\infty 0\,dx \leq \int_3^\infty f(x)\,dx &\leq \int_3^\infty x^{-3}\,dx \\ 0 \leq \int_3^\infty f(x)\,dx &\leq \lim_{a\to\infty} \int_3^a x^{-3}\,dx \\ 0 \leq \int_3^\infty f(x)\,dx &\leq \lim_{a\to\infty} \left[\frac{-1}{2x^2}\right]_3^a \\ 0 \leq \int_3^\infty f(x)\,dx &\leq \lim_{a\to\infty} \frac{-1}{2a^2} + \frac{1}{18} \\ 0 \leq \int_3^\infty f(x)\,dx &\leq \frac{1}{18}. \end{aligned}$$

18. (7 points) Using what you learned about inverse functions, find a formula for the derivative $\frac{d}{dx}\sinh^{-1}(x)$. Your answer should be expressed in a way that does not use hyperbolic functions or their inverses.

Solution: We can begin by taking the derivative of both sides of our basic inverse function relationship, e.g., that

$$\sinh\left(\sinh^{-1}(x)\right) = x.$$

Taking the derivative of both sides, and remembering to use the chain rule, we get that

$$\cosh\left(\sinh^{-1}(x)\right) \cdot \frac{d}{dx}\sinh^{-1}(x) = 1 \implies \frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\cosh\left(\sinh^{-1}(x)\right)}. \tag{1}$$

To get rid of the hyperbolic functions we now need to use that $\cosh^2(x) - \sinh^2(x) = 1$ from the formula sheet. Rearranging this we learn that $\cosh(x) = \sqrt{1+\sinh^2(x)}$. As a result we have that

$$\cosh\left(\sinh^{-1}(x)\right) = \sqrt{1+\sinh^2(\sinh^{-1}(x))} = \sqrt{1+x^2}$$

Plugging this into (1) we learn that

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$
- $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\ dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

Derivatives

- $\dfrac{d}{dx}(\sinh x) = \cosh x$ $\quad$ $\dfrac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\cos^2 x + \sin^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$