Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Determine if the following series converge or diverge. **If the series converges, also compute the sum**. You must show all of your work and support your conclusions.

(a) (7 points) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$

Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n} &= \sum_{n=0}^{\infty} \frac{2}{2} \cdot \frac{2^{n-1}}{3^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{2^n}{3^n} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ &= \frac{1}{2}\left(\frac{1}{1-2/3}\right) = \frac{3}{2}\end{aligned}$$

And so the series is a geometric with $|r| = 2/3 < 1$ and so it converges to $3/2$ as shown above

(b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n-1}$

Solution: Compare with $b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{2}{n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n}{n-2} = 2$$

So since b_n diverges by the p-series test, $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n-1}$ also diverges by the limit comparison test.

2. Determine if the following series converge or diverge. You must show all of your work and justify your use of any series convergence tests.

(a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^5}$

Solution: Consider the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)^5}}{\frac{5^n}{n^5}} &= \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)^5} \cdot \frac{n^5}{5^n} \\
&= \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \cdot \frac{n^5}{(n+1)^5} \\
&= \lim_{n\to\infty} 5 \cdot \left(\frac{n}{n+1}\right)^5 \\
&= 5 \cdot (1)^5 = 5 > 1
\end{aligned}$$

So by the ratio test $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^5}$ diverges.

(b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{2n}{n^2 - n^3}$

Solution: Notice that when $n \geq 2$ all the terms in the series is negative. So write

$$\sum_{n=2}^{\infty} \frac{2n}{n^2 - n^3} = -\sum_{n=2}^{\infty} \frac{2n}{n^3 - n^2}.$$

The sum is now a series with only positive terms. We can then apply the limit comparison test against $1/n^2$:

$$\lim_{n\to\infty} \frac{\frac{2n}{n^3-n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2}{1 - \frac{1}{n}} = 2$$

which is positive and finite. Since $\sum 1/n^2$ converges by the p-series test with $p = 2 > 1$, so must the series $\sum_{n=2}^{\infty} \frac{2n}{n^3-n^2}$, and thus also the original series.

3. For each of the following functions, find its 3rd degree Taylor polynomial centered at the given a.

(a) (7 points) $f(x) = \sin(-5x)$, centered at $a = 0$.

Solution:

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \sin(-5x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (-5x)^{2n+1}}{(2n+1)!} \\ &= (-5x) + \frac{(-1)(-5x)^3}{(3)!} + \cdots = \boxed{-5x + \frac{125}{6}x^3} + \cdots \end{aligned}$$

(b) (7 points) $g(x) = xe^x$, centered at $a = 0$.

Solution:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ xe^x &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \\ &= \boxed{x + x^2 + \frac{x^3}{2}} + \cdots \end{aligned}$$

4. (7 points) Find the Maclaurin series (Taylor series centered at $a = 0$) representation of $f(x) = \dfrac{x}{1 + x^2/2}$. Express your answer in sigma notation.

Solution:

$$
\begin{aligned}
f(x) &= x\left(\frac{1}{1-\left(\frac{-x^2}{2}\right)}\right) \\
&= x\left(\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n}
\end{aligned}
$$

5. (7 points) Find the open interval of convergence for the power series $\displaystyle\sum_{n=5}^{\infty}\frac{(2x-3)^n}{n^2-2}$. (You do not have to test the end points for convergence.)

Solution: According to the ratio test

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(2x-3)^{n+1}}{(n+1)^2-2}}{\frac{(2x-3)^n}{n^2-2}}\right| &= \lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(2x-3)^n}\cdot\frac{n^2-2}{(n+1)^2-2}\right| \\
&= |2x-3|\cdot\lim_{n\to\infty}\frac{n^2-2}{(n+1)^2-2} \\
&= |2x-3|
\end{aligned}
$$

only converges if $|2x-3| < 1$. This is equivalent to the interval $(1, 2)$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{n(-1)^n}{\ln(n)+1}$?

   A. The **ratio test** shows that the series converges.

   B. The **integral test** shows that the series diverges.

   C. The **alternating series test** shows that the series converges.

   D. The **$\mathbf{n^{th}}$-term test** shows that the series diverges.

   E. None of the above are true.

7. (4 points) Find the limit of the **sequence** a_n where the n^{th} term is given by $a_n = \frac{\arctan n}{4n}$.

   A. 0

   B. $\frac{1}{4}$

   C. $\frac{\pi}{8}$

   D. 4

   E. The sequence diverges

8. (4 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1) \sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}+1} \quad \text{and} \quad (2) \sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}}$$

   A. (1) is absolutely convergent; (2) is divergent.

   B. (1) is conditionally convergent; (2) is divergent.

   C. (1) is absolutely convergent; (2) is conditionally convergent.

   D. (1) is divergent; (2) is conditionally convergent.

   E. Both (1) and (2) are conditionally convergent.

9. (4 points) The Taylor series of the function $f(x)$, centered at $a = 1$, is given by $\sum_{n=0}^{\infty} \frac{n^2-1}{n!}(x-1)^n$. What is the value of the second derivative $f''(1)$?

A. 0

B. 1

C. 3/2

D. 3

E. 4

10. (4 points) The first 3 nonzero terms of the Taylor series, centered at $a = 0$, for $f(x) = \frac{x^3}{1-x^2}$ are

A. $x^3 - x^5 + x^7$

B. $x^3 + x^5 + x^7$

C. $\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$

D. $\frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

E. $1 + x^2 + x^3$

11. (4 points) What is the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{(3x-1)^n}{n}$?

A. 4/3

B. 1

C. 2/3

D. 1/3

E. 0

12. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{4\ln(n)}{n^3}$?

A. By the ratio test, the series converges.

B. By the ratio test, the series diverges.

C. The ratio test is inconclusive for this series, but the series converges by another test.

D. The ratio test is inconclusive for this series, but the series diverges by another test.

E. None of the above are true.

13. (4 points) What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{n \cdot x^n}{n!}$?

A. 0

B. 1

C. 2

D. 4

E. ∞

14. (4 points) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 2$. Which of the following statements must also be true?

A. The power series also converges at $x = 1$.

B. The power series also converges at $x = -1$.

C. The power series also converges at $x = -2$.

D. Only (A) and (B) must hold.

E. All of (A), (B), and (C) must hold.

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

15. (4 points) Which statement about the series $\displaystyle\sum_{n=2}^{\infty} \frac{500}{n\,(\ln n)}$ is true?

   A. It converges by the integral test.

   B. It diverges by a comparison to the divergent series $\displaystyle\sum_{n=2}^{\infty} \frac{500}{n}$.

   C. It converges by a comparison to the convergent series $\displaystyle\sum_{n=2}^{\infty} \frac{500}{n}$.

   D. It diverges by the p-series test with $p = 1$.

   E. None of the above.

16. (4 points) Suppose $\{a_n\}_{n=0}^{\infty}$ is a convergent sequence. **Answer both of these True/False Questions**.

- The series $\sum (-1)^n a_n$ always converges. **FALSE**
- The series $\sum (-1)^n \, |a_n - \lim_{n\to\infty} a_n|$ always converges. **FALSE**

17. (6 points) Evaluate $\displaystyle\lim_{x\to 0^+} \frac{e^{x^{100}} - 1}{x^{10}\sin(x^{90}/2)}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{e^{x^{100}} - 1}{x^{10}\sin(x^{90}/2)} &= \lim_{x\to 0^+} \frac{\sum_{n=0}^{\infty} \frac{(x^{100})^n}{n!} - 1}{x^{10}\left(\sum_{n=0}^{\infty} \frac{(x^{90}/2)^{2n+1}}{(2n+1)!}\right)} \\
&= \lim_{x\to 0^+} \frac{(1 + x^{100} + \cdots) - 1}{x^{10}\,(x^{90}/2 + \cdots)} \\
&= \lim_{x\to 0^+} \frac{x^{100} + \cdots}{x^{100}/2 + \cdots} = 2
\end{aligned}$$