

Name: _____

Section: _____ Recitation Instructor: _____

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the
above instructions and statements
regarding academic honesty: _____

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Determine if the following series converge or diverge. **If the series converges, also compute the sum.** You must show all of your work and support your conclusions.

(a) (7 points) $\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$

(b) (7 points) $\sum_{n=2}^{\infty} \frac{2}{n-1}$

2. Determine if the following series converge or diverge. You must show all of your work and justify your use of any series convergence tests.

(a) (7 points) $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$

(b) (7 points) $\sum_{n=2}^{\infty} \frac{2n}{n^2 - n^3}$

3. For each of the following functions, find its 3rd degree Taylor polynomial centered at the given a .

(a) (7 points) $f(x) = \sin(-5x)$, centered at $a = 0$.

(b) (7 points) $g(x) = xe^x$, centered at $a = 0$.

4. (7 points) Find the Maclaurin series (Taylor series centered at $a = 0$) representation of $f(x) = \frac{x}{1 + x^2/2}$. Express your answer in sigma notation.

5. (7 points) Find the open interval of convergence for the power series $\sum_{n=5}^{\infty} \frac{(2x - 3)^n}{n^2 - 2}$. (You do not have to test the end points for convergence.)

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{n(-1)^n}{\ln(n) + 1}$?
- A. The **ratio test** shows that the series converges.
 - B. The **integral test** shows that the series diverges.
 - C. The **alternating series test** shows that the series converges.
 - D. The **nth-term test** shows that the series diverges.
 - E. None of the above are true.
7. (4 points) Find the limit of the **sequence** a_n where the nth term is given by $a_n = \frac{\arctan n}{4n}$.
- A. 0
 - B. $\frac{1}{4}$
 - C. $\frac{\pi}{8}$
 - D. 4
 - E. The sequence diverges
8. (4 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:
- (1) $\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2} + 1}$ and (2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}}$
- A. (1) is absolutely convergent; (2) is divergent.
 - B. (1) is conditionally convergent; (2) is divergent.
 - C. (1) is absolutely convergent; (2) is conditionally convergent.
 - D. (1) is divergent; (2) is conditionally convergent.
 - E. Both (1) and (2) are conditionally convergent.

9. (4 points) The Taylor series of the function $f(x)$, centered at $a = 1$, is given by $\sum_{n=0}^{\infty} \frac{n^2 - 1}{n!} (x - 1)^n$.

What is the value of the second derivative $f''(1)$?

- A. 0
- B. 1
- C. $3/2$
- D. 3
- E. 4

10. (4 points) The first 3 nonzero terms of the Taylor series, centered at $a = 0$, for $f(x) = \frac{x^3}{1 - x^2}$ are

- A. $x^3 - x^5 + x^7$
- B. $x^3 + x^5 + x^7$
- C. $\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$
- D. $\frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$
- E. $1 + x^2 + x^3$

11. (4 points) What is the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{(3x - 1)^n}{n}$?

- A. $4/3$
- B. 1
- C. $2/3$
- D. $1/3$
- E. 0

12. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{4 \ln(n)}{n^3}$?

- A. By the ratio test, the series converges.
- B. By the ratio test, the series diverges.
- C. The ratio test is inconclusive for this series, but the series converges by another test.
- D. The ratio test is inconclusive for this series, but the series diverges by another test.
- E. None of the above are true.

13. (4 points) What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{n \cdot x^n}{n!}$?

- A. 0
- B. 1
- C. 2
- D. 4
- E. ∞

14. (4 points) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 2$. Which of the following statements must also be true?

- A. The power series also converges at $x = 1$.
- B. The power series also converges at $x = -1$.
- C. The power series also converges at $x = -2$.
- D. Only (A) and (B) must hold.
- E. All of (A), (B), and (C) must hold.

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

15. (4 points) Which statement about the series $\sum_{n=2}^{\infty} \frac{500}{n(\ln n)}$ is true?

A. It converges by the integral test.

B. It diverges by a comparison to the divergent series $\sum_{n=2}^{\infty} \frac{500}{n}$.

C. It converges by a comparison to the convergent series $\sum_{n=2}^{\infty} \frac{500}{n}$.

D. It diverges by the p -series test with $p = 1$.

E. None of the above.

16. (4 points) Suppose $\{a_n\}_{n=0}^{\infty}$ is a convergent sequence. **Answer both of these True/False Questions.**

• The series $\sum (-1)^n a_n$ always converges. **T** **F**

• The series $\sum (-1)^n |a_n - \lim_{n \rightarrow \infty} a_n|$ always converges. **T** **F**

17. (6 points) Evaluate $\lim_{x \rightarrow 0^+} \frac{e^{x^{100}} - 1}{x^{10} \sin(x^{90}/2)}$.

DO NOT WRITE BELOW THIS LINE.

| Page | Points | Score |
|--------|--------|-------|
| 2 | 14 | |
| 3 | 14 | |
| 4 | 14 | |
| 5 | 14 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 12 | |
| 9 | 14 | |
| Total: | 106 | |

No more than 100 points may be earned on the exam.

FORMULA SHEET PAGE 1

Integrals

- **Volume:** Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x -axis, then the volume of S is given by

$$V = \int_a^b A(x) dx$$

- **Work:** Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x) dx$$

- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int \tan x dx = \ln|\sec x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$
- **Integration by Parts:**

$$\int u dv = uv - \int v du$$

- **Arc Length Formula:**

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Derivatives

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$

- **Inverse Trigonometric Functions:**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- **Hyperbolic Functions**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \quad \operatorname{coth}(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

FORMULA SHEET PAGE 2

Series

- **n th term test for divergence:** If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The p -series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

(i) If $\int_1^{\infty} f(x) dx$ is convergent,
then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x) dx$ is divergent,
then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

- $0 < b_{n+1} \leq b_n$ for all n
- $\lim_{n \rightarrow \infty} b_n = 0$

then the series is convergent.

- **The Ratio Test**

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series:** $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

- **Some Power Series**

$$\circ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty$$

$$\circ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R = \infty$$

$$\circ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad R = \infty$$

$$\circ \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R = 1$$

$$\circ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad R = 1$$