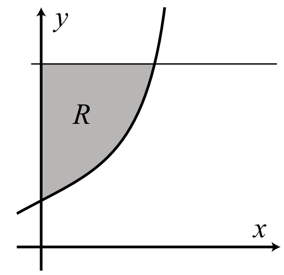
Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) Let R be the region in the first quadrant below the line $y = 3$ and above the curve $y = e^x$, (shown in the picture). Find the volume of the solid formed by revolving R about the x-axis.

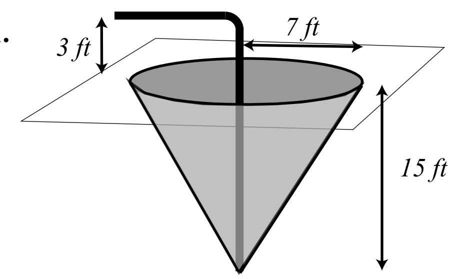


Solution: Intersection of $y = 3$ and $y = e^x$ happens at $e^x = 3 \implies x = \ln 3$. So now we evaluate

$$\pi \int_0^{\ln 3} 9 - e^{2x}\, dx = \pi \left[9x - \frac{1}{2}e^{2x}\right]_0^{\ln 3} = \pi[9\ln 3 - \tfrac{1}{2}\cdot 9 + \tfrac{1}{2}] = \boxed{\pi[9\ln(3) - 4]}$$

2. (7 points) An underground tank is a cone with radius 7 feet and height 15 feet. The tank is full of water (water weighs 62.5 lb/ft^3). How much work is required to empty the tank by pumping the water to a height of 3 ft above ground level?

Write your answer as an integral, but **do not evaluate the integral.**



Solution: Measure y from the bottom. The radius and height of the cone are related through $r = \frac{7}{15}y$. Distance to pump is $18 - y$ and volume of slice is $\pi\left(\frac{7}{15}y\right)^2\, dy$. So

$$\text{Work} = \int_0^{15} \pi\left(\frac{7}{15}y\right)^2 \cdot 62.5 \cdot (18 - y)\, dy \quad \text{ft-lbs.}$$

Remark: *if labeling with y from ground level, then* $r(y) = 7 - \frac{7}{15}y$, *and distance pumped is* $y + 3$. *So the resulting integral would be*

$$\int_0^{15} \pi\left(7 - \frac{7}{15}y\right)^2 \cdot 62.5 \cdot (3 + y)\, dy.$$

3. (7 points) Differentiate the function $f(x) = \big(\ln x\big)^{\cosh(x)}$.

Solution: Log differentiation:

$$\frac{1}{y} \cdot y' = \frac{d}{dx}\left[\cosh(x) \cdot \ln(\ln(x))\right] = \left[\sinh(x)\ln(\ln(x)) + \cosh(x)\frac{1}{\ln(x)}\frac{1}{x}\right]$$

so

$$y' = (\ln x)^{\cosh(x)}\left[\sinh(x)\ln(\ln(x)) + \cosh(x)\frac{1}{\ln(x)}\frac{1}{x}\right]$$

Solution: Change of base:

$$f(x) = e^{\ln(\ln(x))\cosh(x)}$$

so

$$f'(x) = e^{\ln(\ln(x))\cosh(x)}\left[\cosh(x) \cdot \ln(\ln(x))\right]' = e^{\ln(\ln(x))\cosh(x)}\left[\sinh(x)\ln(\ln(x)) + \cosh(x)\frac{1}{\ln(x)}\frac{1}{x}\right]$$

4. (7 points) Use partial fractions to find $\displaystyle\int \frac{x+7}{x^2-x-2}\,dx$.

Solution: $x^2 - x - 2 = (x-2)(x+1)$. So $\dfrac{x+7}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$. Multiply through

$$x + 7 = A(x+1) + B(x-2).$$

Solve (cover up or linear system) $\implies A = 3, B = -2$.

$$\int \frac{3}{x-2} - \frac{2}{x+1}\,dx = \boxed{3\ln(|x-2|) - 2\ln(|x+1|) + C}.$$

5. (6 points) Evaluate the definite integral $\displaystyle\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\, dx$.

Solution: u substitution: $u = \ln x$, $du = \frac{1}{x}\, dx$. So

$$\int \frac{1}{x\sqrt{\ln x}}\, dx = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C$$

so therefore

$$\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\, dx = 2\sqrt{\ln x}\Big|_e^{e^2} = 2\sqrt{lne^2} - 2\sqrt{\ln e} = \boxed{2\sqrt{2} - 2}$$

6. (a) (2 points) Find the derivative of $y = x\ln(x) - x$.

Solution:

$$y' = \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x).$$

(b) (6 points) Integrate $\displaystyle\int (\ln x)^2\, dx$.

Solution: Integrate by parts $\begin{array}{ll} u = (\ln x)^2 & du = 2\ln(x)\frac{1}{x}dx \\ v = x & dv = dx \end{array}$

$$= x(\ln x)^2 - \int 2\ln(x)\, dx$$

The second integral we can use part (a):

$$= \boxed{x(\ln x)^2 - 2(x\ln(x) - x) + C}$$

Remark: *alternatively one can integrate by parts a second time.*

7. (7 points) For the function $y = \sin^{-1}(x)$, derive the formula $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ using either implicit differentiation or the general formula for $(f^{-1})'$. *Show all work and make your reasoning clear.*

Solution: Implicit differentiation:

$$\begin{aligned} y &= \sin^{-1}(x) \\ \sin(y) &= x \\ \cos(y) \cdot y' &= 1 \\ y' &= \frac{1}{cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

(*Students should show some work evaluating* $\cos(\sin^{-1}(x))$ *using either trig identity* $\cos^2(y) = 1 - \sin^2(y)$ *or triangle drawing.*)

Solution: Inverse function formula:

$$(\sin^{-1})'(x) = \frac{1}{(\sin)'(y)} \qquad \text{when } x = \sin(y).$$

So

$$(\sin^{-1})'(x) = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

(Student should show some work evaluating $\cos(\sin^{-1}(x))$ using either trig identity $\cos^2(y) = 1 - \sin^2(y)$ or triangle drawing.)

8. (7 points) Integrate $\displaystyle\int \frac{1}{\left(9+x^2\right)^{3/2}}\, dx$.

Solution: Trig substitution: $x = 3\tan(\theta)$, $dx = 3\sec^2(\theta)d\theta$, and $\sqrt{9+x^2} = 3\sec(\theta)$.

$$= \int \frac{1}{(3\sec(\theta))^3} \cdot 3\sec^2(\theta)\ d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\ d\theta = \frac{1}{9}\int \cos\theta\ d\theta = \frac{1}{9}\sin\theta + C.$$

If $\tan\theta = x/3$, then $\sin\theta = x/\sqrt{3^2+x^2}$ (draw a triangle, or use that $\sin\theta = \tan\theta/\sec\theta$). So

$$\int \frac{1}{\left(9+x^2\right)^{3/2}}\ dx = \boxed{\frac{1}{9}\frac{x}{\sqrt{9+x^2}} + C}.$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

9. (4 points) The derivative of $y = \cosh(x^2) + \sinh(2x)$ is:

   A. $2x\ \sinh(x^2) + 2\cosh(2x)$

   B. $-2x\ \sinh(x^2) + 2\cosh(2x)$

   C. $2x\ \sinh(2x) - 2\cosh(x^2)$

   D. $2x\ \sinh(x^2) - 2\cosh(2x)$

   E. $2x\ \sinh(2x) + 2\cosh(x^2)$

10. (4 points) The function $f(x) = \int_1^{e^x} \frac{1}{t}\,dt$ is equal to

    A. $e^x - 1$ B. $\frac{1}{e^x} - 1$ C. $1 - \frac{1}{e^{2x}}$ D. x E. $x - 1$

11. (4 points) What is the solution $y(x)$ of the initial value problem $y' = \frac{x^2+1}{2y}$ with $y(0) = 1$?

    A. $y = e^{\frac{1}{6}x^3 + \frac{1}{2}x}$

    B. $y = \sqrt{\frac{1}{3}x^3 + x + 1}$

    C. $y = \frac{1}{6}x^3 + \frac{1}{2}x + 1$

    D. $y = 1 - \sqrt{x^3 + x}$

    E. $y = \frac{1}{2}x^2 + 1$

12. (4 points) Evaluate $\int_0^{\pi/4} \tan^6(x)\sec^4(x)\,dx$.

A. $\frac{1}{9}+\frac{1}{7}$ B. $\frac{1}{8}+\frac{1}{6}$ C. $\frac{1}{7}-\frac{1}{5}$ D. $\frac{1}{7}+\frac{1}{5}$ E. 0

13. (4 points) What is the form of the partial fraction decomposition of $\frac{1}{(x^2+1)(x-3)^2}$?

A. $\frac{Ax+B}{x^2+1}+\frac{C}{x-3}$

B. $\frac{Ax+B}{x^2+1}+\frac{C}{(x-3)^2}$

C. $\frac{A}{x^2+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

D. $\frac{A}{x+1}+\frac{B}{x^2+1}+\frac{C}{x-3}+\frac{D}{(x-3)^2}$

E. $\frac{Ax+B}{x^2+1}+\frac{C}{x-3}+\frac{D}{(x-3)^2}$

14. (4 points) Compute the integral $\int \frac{\sin x}{3+\cos x}\,dx$.

A. $\ln(3+\cos x)+C$

B. $-\ln(3+\cos x)+C$

C. $\ln(3+\sec x)+C$

D. $\frac{1}{3}x+\ln(|\sec(x)|)+C$

E. $-\frac{1}{3}\cos(x)+\ln(|\sec(x)|)+C$

15. (4 points) Is the improper integral $\int_1^\infty \frac{2+\sin(x)}{x^2}\,dx$ convergent?

A. It is convergent because $0 \le \frac{1}{x^2} \le \frac{2+\sin(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

B. It is convergent because $0 \le \frac{2+\sin(x)}{x^2} \le \frac{3}{x^2}$ and $\int_1^\infty \frac{3}{x^2}\,dx$ is convergent.

C. It is divergent because $0 \le \frac{2+\sin(x)}{x^2} \le \frac{3}{x^2}$ and $\int_1^\infty \frac{3}{x^2}\,dx$ is divergent.

D. It is divergent because $0 \le \frac{1}{x^2} \le \frac{2+\sin(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ is divergent.

E. None of the above; the comparison theorem for improper integrals cannot be applied to this integral.

16. (4 points) Evaluate $\lim_{x\to\infty} \frac{\ln(3x^2)}{\ln(5x+1)}$.

A. $\frac{3}{5}$ B. $\frac{6}{5}$ C. 2 D. $\frac{\ln(6)}{\ln(5)}$ E. The limit does not exist

17. (4 points) The rate of decay of a radioactive material is proportional to the amount of that material present. If it takes 5 years for one third of the material to decay, how many years does it take for half of the material to decay?

A. $\frac{5\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)}$ B. $\frac{\ln\left(\frac{2}{3}\right)}{5}$ C. $\frac{\ln\left(\frac{2}{3}\right)}{5\ln\left(\frac{1}{2}\right)}$ D. $\frac{5\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$ E. $5\ln\left(\frac{3}{4}\right)$

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

18. (8 points) Find the integral $\int \left(\sqrt{\sin(2x)} \;-\; \cos(2x)\right)^2 \, dx$

Solution: First expand the square to get

$$= \int \sin(2x) - 2\sqrt{\sin(2x)}\cos(2x) + \cos^2(2x) \, dx$$

The three terms we integrate separately:

$$\begin{aligned}
\int \sin(2x) \, dx &= -\frac{1}{2}\cos(2x) + C \\
\int -2\sqrt{\sin(2x)}\cos(2x) \, dx &= -\int \sqrt{u} \, du \qquad\qquad (u = \sin(2x), du = 2\cos(2x)dx) \\
&= -\frac{2}{3}u^{3/2} + C \\
&= -\frac{2}{3}\sin^{3/2}(2x) + C \\
\int \cos^2(2x) \, dx &= \int \frac{1}{2}(1 + \cos(4x)) \, dx \\
&= \frac{1}{2}x + \frac{1}{8}\sin(4x) + C
\end{aligned}$$

So, final answer is:

$$\boxed{-\frac{1}{2}\cos(2x) - \frac{2}{3}\sin^{3/2}(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x) + C}$$

19. (6 points) Prove that the curves $y = \frac{\pi}{6} - \sin^{-1}(\frac{x}{2})$ and $y = \frac{\sqrt{3}}{2}\left(e^{x^2-1} - 1\right)$ intersect at right angles at the point $(1, 0)$.

Solution: Two curves intersect at right angles if their slopes multiply to equal -1. (From Calc I.)

Curve 1:

$$y' = -\frac{1}{\sqrt{1 - \frac{x^2}{4}}} \cdot \frac{1}{2} \implies y'(1) = -\frac{1}{\sqrt{3}}.$$

Curve 2:

$$y' = \frac{\sqrt{3}}{2}e^{x^2-1} \cdot 2x \implies y'(1) = \sqrt{3}.$$

Indeed, the product of their slopes multiply to equal -1.