Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) Find a parametrization for line of intersection between the planes $x+y+z=1$ and $2x+z=3$

Solution:

$$\begin{aligned}\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= (1-0)\mathbf{i} - (1-2)\mathbf{j} + (0-2)\mathbf{k} \\ &= \langle 1, 1, -2 \rangle\end{aligned}$$

and we need a point of intersection. There are lots of choices. Setting $x = 0$ we get $z = 3$ and then $y = -2$. Giving us the final answer

$$\begin{aligned}\mathbf{r}(t) &= \langle 1, 1, -2 \rangle\, t + \langle 0, -2, 3 \rangle & t \in (-\infty, \infty) \\ &= \langle t, t-2, -2t+3 \rangle & t \in (-\infty, \infty)\end{aligned}$$

(remember there are infinitely many correct parametrizations for any curve.)

2. (7 points) What is the distance between the origin and the plane $x - y + 3z = 2$?

Solution: One point on the plane is $R(2, 0, 0)$. $\mathbf{n} = \langle 1, -1, 3 \rangle$. So $\overrightarrow{OR} = \langle 2, 0, 0 \rangle$ and therefore the distance between the plane and the origin is

$$d = \frac{|\mathbf{n} \cdot \overrightarrow{OR}|}{|\mathbf{n}|} = \frac{|2|}{\sqrt{1+1+9}} = \boxed{\frac{2}{\sqrt{11}}}$$

3. An object moves along the curve $\mathbf{r}(t) = \langle 3, 1 - t^2, 5 + t^3 \rangle$ for time $t \geq 0$.

(a) (5 points) What is the speed of the object at time $t = 2$?

Solution:

$$\begin{aligned} |\mathbf{r}'(t)| &= |\left\langle 0, -2t, 3t^2 \right\rangle| \\ |\mathbf{r}'(2)| &= |\langle 0, -4, 12 \rangle| \\ &= \sqrt{0 + 16 + 144} = \boxed{\sqrt{160}} \end{aligned}$$

(b) (9 points) What is the length of the curve from $t = 0$ to $t = 2$?

Solution:

$$|\mathbf{r}'(t)| = |\left\langle 0, -2t, 3t^2 \right\rangle| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2} \qquad (\text{since } t \geq 0)$$

and

$$\begin{aligned} \int t\sqrt{4 + 9t^2}\ dt &= \int \sqrt{u}\ \frac{du}{18} \qquad (u = 4 + 9t^2 \text{ and } du = 18tdt) \\ &= \frac{1}{27} u^{3/2} \\ &= \frac{1}{27} (4 + 9t^2)^{3/2} \end{aligned}$$

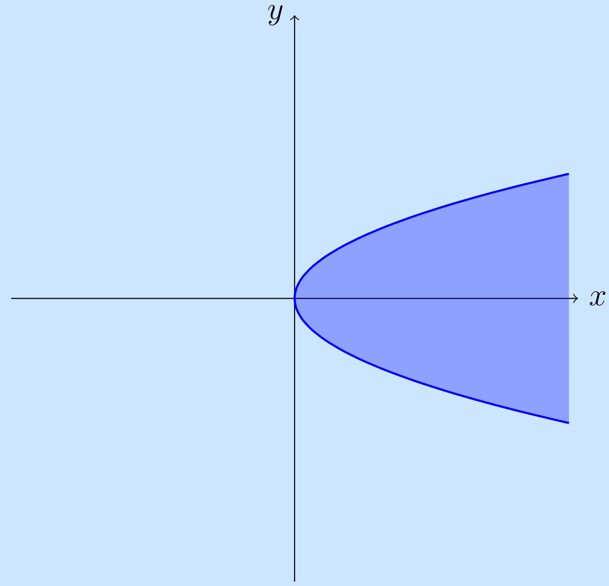
so therefore

$$L = \int_0^2 |\mathbf{r}'(t)|\ dt = \int_0^2 t\sqrt{4 + 9t^2}\ dt = \boxed{\frac{1}{27}\left(40^{3/2} - 8\right)}$$

4. Consider the function $f(x, y) = \sqrt{x - y^2}$

(a) (5 points) Sketch the domain of f.

Solution: $x - y^2 \geq 0 \implies x \geq y^2$



(b) (6 points) Sketch the level curves of f for $k = 0$, $k = 1$, and $k = 2$.

Solution: $\underline{k = 0}$

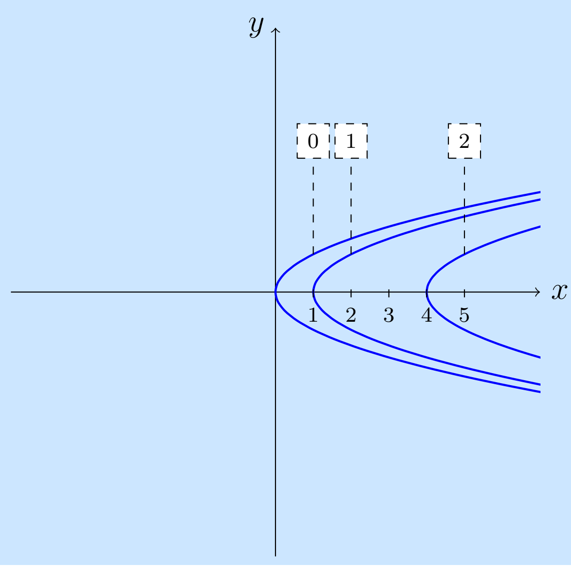
$$\begin{aligned} \sqrt{x - y^2} &= 0 \\ x - y^2 &= 0 \\ x &= y^2 \end{aligned}$$

$\underline{k = 1}$

$$\begin{aligned} \sqrt{x - y^2} &= 1 \\ x - y^2 &= 1 \\ x &= y^2 + 1 \end{aligned}$$

$\underline{k = 2}$

$$\begin{aligned} \sqrt{x - y^2} &= 2 \\ x - y^2 &= 4 \\ x &= y^2 + 4 \end{aligned}$$



(c) (3 points) What is the range of f?

Solution: $[0, \infty)$

5. (7 points) Evaluate the limit or show that it does not exist: $\lim\limits_{\substack{x\to 3\\ y\to -3}} \dfrac{x^2-y^2}{x+y}$

Solution:

$$\begin{aligned}\lim_{\substack{x\to 3\\ y\to -3}} \frac{x^2-y^2}{x+y} &= \lim_{\substack{x\to 3\\ y\to -3}} \frac{\cancel{(x+y)}(x-y)}{\cancel{x+y}}\\ &= \lim_{\substack{x\to 3\\ y\to -3}} x-y = 3-(-3) = \boxed{6}\end{aligned}$$

6. (7 points) Evaluate the limit or show that it does not exist: $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{xy}{x+y-1}$

Solution: Consider the following paths

Path 1: $x=1, y\to 0$

$$\lim_{y\to 0} \frac{(1)y}{(1)+y-1} = \lim_{y\to 0} \frac{y}{y} = 1$$

Path 2: $y=0, x\to 1$

$$\lim_{x\to 1} \frac{x(0)}{x+0-1} = \lim_{x\to 1} \frac{0}{x-1} = 0$$

Since the limit approaches different values along different paths we know $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{xy}{x+y-1}$ $\boxed{\text{DNE}}$.

7. Consider the function $f(x,y) = 4x\ln(xy-1)+3$.

(a) (5 points) Calculate the partial derivatives, f_x and f_y.

Solution:

$$f_x = 4\ln(xy-1) + \frac{4xy}{xy-1}$$
$$f_y = \frac{4x^2}{xy-1}$$

(b) (5 points) Use your answer in part (a) to find the linearization of f at $(1,2)$.

Solution:

$$f_x(1,2) = 0 + \frac{8}{1} = 8$$
$$f_y(1,2) = \frac{4}{1} = 4$$
$$f(1,2) = 3$$

so

$$L(x,y) = 3 + 8(x-1) + 4(y-2)$$
$$L(x,y) = 8x + 4y - 13$$

(c) (4 points) Use your answer in part (b) to approximate $f(1.2, 1.9)$.

Solution:

$$\begin{aligned} L(1.2, 1.9) &= 3 + 8(1.2-1) + 4(1.9-2) \\ &= 3 + 8(.2) + 4(-.1) \\ &= 3 + 1.6 - .4 \\ &= 4.2 \end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Find the radius and center for the sphere: $x^2 + (y+1)^2 + z^2 + 6z = 7$

   A. center: $(0, -1, 6)$, radius: $\sqrt{7}$

   B. center: $(0, 1, 3)$, radius: $\sqrt{7}$

   C. center: $(0, -1, -3)$, radius: $\sqrt{7}$

   D. center: $(0, -1, -3)$, radius: 4

   E. center: $(0, 1, 3)$, radius: 4

9. (4 points) Consider the vectors **a** and **b** shown to the right in $\mathbb{R}^2$. Which of the following vectors could be $\text{proj}_{\mathbf{b}}\mathbf{a}$?

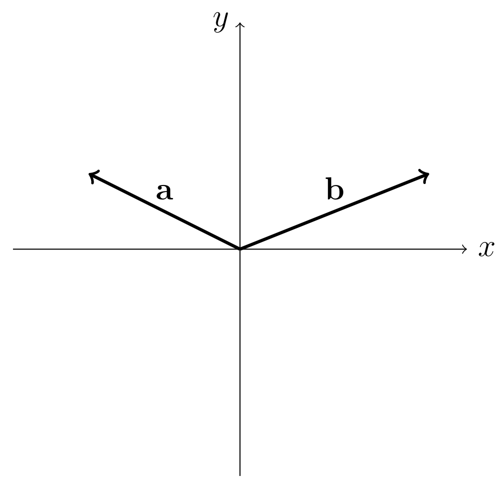
   A. $\langle 2, 1 \rangle$

   B. $\langle -2, 1 \rangle$

   C. $\langle -2, -1 \rangle$

   D. $\langle 2, -1 \rangle$

   E. $\langle 0, 0 \rangle$



10. (4 points) Determine what quadric surface this equation represents: $y^2 + z = x^2$

    A. Hyperboloid of One Sheet

    B. Elliptic Paraboloid

    C. Hyperbolic Paraboloid

    D. Cone

    E. Hyperboloid of Two Sheets

11. (4 points) Find the area of the parallelogram determined by $\mathbf{i} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$

A. 4
B. $\sqrt{14}$
C. $\sqrt{11}$
D. $\sqrt{10}$
E. 3

12. (4 points) Which of the following is a parametrization for the full curve of intersection between $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2$

A. $\mathbf{r}(t) = \left\langle t, \sqrt{1 - t^2}, 2 \right\rangle$ with $t \in [0, 2\pi]$
B. $\mathbf{r}(t) = \left\langle \sqrt{1 - t^2}, t, 2 \right\rangle$ with $t \in [0, 2\pi]$
C. $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$ with $t \in [0, 2\pi]$
D. $\mathbf{r}(t) = \langle \cos(t), \sin(t), -2 \rangle$ with $t \in [0, 2\pi]$
E. $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ with $t \in [0, 2\pi]$

13. (4 points) Which of the following satisfy: $\mathbf{r}'(t) = \langle 2t, 3, -\sin(t) \rangle$ and $\mathbf{r}(0) = \langle 1, 2, 3 \rangle$

A. $\mathbf{r}(t) = \langle t^2, 3t, \cos(t) \rangle$
B. $\mathbf{r}(t) = \langle t^2 + 1, 3t + 2, \cos(t) + 3 \rangle$
C. $\mathbf{r}(t) = \langle t^2 + 1, 3t + 2, -\cos(t) + 3 \rangle$
D. $\mathbf{r}(t) = \left\langle t^2 + 1, 3t + 2, \cos(t) + 2 \right\rangle$
E. $\mathbf{r}(t) = \langle 2, 0, -\cos(t) \rangle$

14. (4 points) Consider the function $f(x,y) = \dfrac{2x}{y}$. Calculate f_{xy}

A. $f_{xy} = \dfrac{2}{y}$

B. $f_{xy} = \dfrac{-2}{y^2}$

C. $f_{xy} = \dfrac{-2x}{y^2}$

D. $f_{xy} = 2$

E. $f_{xy} = 0$

15. (4 points) Suppose $z = x^2 y$ where x and y are differentiable functions of t with

$$x(1) = 3, \quad x'(1) = 2, \quad y(1) = -4, \quad y'(1) = 5$$

Which of the following is equivalent to $\left.\dfrac{dz}{dt}\right|_{t=1}$?

A. 6

B. -14

C. -15

D. -3

E. -84

16. (4 points) Find the unit tangent vector of $\mathbf{r}(t) = 2t\mathbf{i} + 3\mathbf{j} - \cos(t)\mathbf{k}$ at $t = 0$.

A. $\langle 0, 3, -1 \rangle$

B. $\left\langle 0, \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$

C. $\langle 2, 0, \sin(t) \rangle$

D. $\langle 2, 0, 0 \rangle$

E. $\langle 1, 0, 0 \rangle$

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	14	
7	12	
8	12	
9	12	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\mathbf{u}||\mathbf{v}| \cos\theta$$
- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\, \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Trigonometry

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$