Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (5 points) Let $f(x,y) = x\,e^{y-2} + y^2 \ln x$. Find ∇f at $P(1,2)$.

Solution:

$$f_x = e^{y-2} + \frac{y^2}{x} \Longrightarrow f_x(P) = 5$$
$$f_y = xe^{y-2} + 2y\ln x \Longrightarrow f_y(P) = 1$$

Thus

$$(\nabla f)_P = 5\,\mathbf{i} + 1\,\mathbf{j}$$

2. (9 points) Let $g(x,y) = 8x^3 - 12xy + y^2$. Find and classify each critical point of g as a local minimum, a local maximum, or a saddle point.

Solution:

i. Find the critical points. Notice that $g_x = 24x^2 - 12y$ and $g_y = 2y - 12x$. So g has a critical points at $P(0,0)$ and $Q(3,18)$.

ii. Now let $D(x,y) = g_{xx}\,g_{yy} - g_{xy}^2 = 96x - 144$. Notice that $g_{xx}(Q) = 144 > 0$ and that

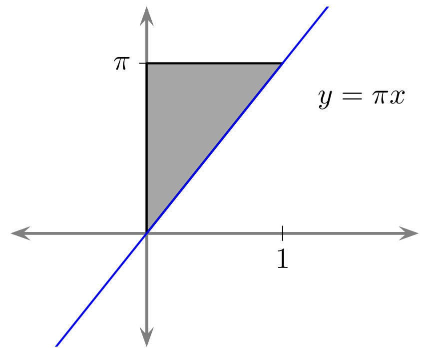
$$D(P) = -144 < 0 \quad \text{and} \quad D(Q) = 288 > 0$$

It follows that g has a local minimum at Q and a saddle point at P.

3. Consider the integral below and answer the questions that follow.

$$\int_0^1 \int_{\pi x}^{\pi} \sin(y^2/\pi)\, dy\, dx$$

(a) (3 points) Sketch the region of integration.



(b) (5 points) Evaluate the integral above by reversing the order of integration.

Solution:

$$= \int_0^{\pi} \sin(y^2/\pi) \int_0^{y/\pi} dx\, dy$$

$$= \frac{1}{\pi} \int_0^{\pi} y \sin(y^2/\pi)\, dy$$

$$= \frac{-1}{2} \cos\left(y^2/\pi\right) \Big|_0^{\pi} = 1$$

4. (6 points) Evaluate the integral below.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{1}{\sqrt{5+x^2+y^2}}\, dy\, dx$$

Solution:

We switch to polar coordinates

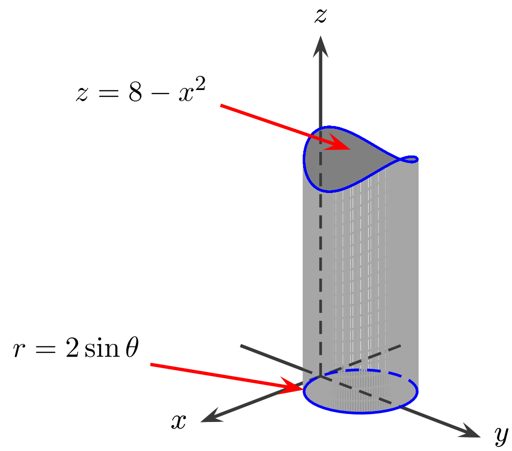
$$= \int_{-\pi/2}^{\pi/2} \int_0^3 \frac{1}{\sqrt{5+r^2}}\, r\, dr\, d\theta$$

$$= \pi \int_0^3 \frac{1}{\sqrt{5+r^2}}\, r\, dr$$

$$= \frac{\pi}{2} \int_5^{14} \frac{1}{\sqrt{u}}\, du = \pi \sqrt{u}\, \Big|_5^{14} = \pi \left(\sqrt{14} - \sqrt{5}\right)$$

5. (6 points) Set up but **Do Not Evaluate** the iterated integral for computing the volume of a region D if D is the right circular cylinder whose base is the disk $r = 2\sin\theta$ (in the xy-plane) and whose top lies on the surface $z = 8 - x^2$.

Solution:



$$\iiint_D dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{8-x^2} r\,dz\,dr\,d\theta$$

Of course, $8 - x^2 = 8 - r^2\cos^2\theta$.

6. (8 points) Find the area of the surface of the cap cut from the paraboloid $z = 12 - x^2 - y^2$ by the cone $z = \sqrt{x^2+y^2}$.

Solution:

Let $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (12 - x^2 - y^2)\,\mathbf{k}$. Then

$$\mathbf{r}_x = \mathbf{i} - 2x\,\mathbf{k}$$
$$\mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$$

So that

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y|^2 = 1 + 4x^2 + 4y^2$$

Thus

$$\begin{aligned} SA &= \iint_{x^2+y^2\le 9} |\mathbf{r}_x \times \mathbf{r}_y|\,dA \\ &= \iint_{x^2+y^2\le 9} \sqrt{1+4x^2+4y^2}\,dA \\ &= 2\pi\int_0^3 \sqrt{1+4r^2}\,r\,dr \\ &= \frac{\pi}{6}\left(37^{3/2} - 1\right) \end{aligned}$$

7. Let $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 - \cos z)\,\mathbf{j} + y\sin z\,\mathbf{k}$ and answer the questions below.

(a) (7 points) Find a function f so that $\nabla f = \mathbf{F}$.

Solution:

Let

$$f(x, y, z) = x^2 y - y\cos z$$

then

$$\nabla f = 2xy\,\mathbf{i} + (x^2 - \cos z)\,\mathbf{j} + y\sin z\,\mathbf{k}$$

(b) (4 points) Let C be any path from $(-2, 1, \pi/2)$ to $(1, 3, \pi)$. Evaluate the integral

$$\int_C 2xy\,dx + (x^2 - \cos z)\,dy + y\sin z\,dz = I$$

Solution:

Observe that

$$df = 2xy\,dx + (x^2 - \cos z)\,dy + y\sin z\,dz$$

It follows that

$$I = \int_{(-2,1,\pi/2)}^{(1,3,\pi)} df = f\,\Big|_{(-2,1,\pi/2)}^{(1,3,\pi)} = 6 - 4$$

(c) (3 points) Calculate $\operatorname{curl}\mathbf{F}$.

Solution:

By part (a), $\mathbf{F}$ is a conservative vector field. It follows that $\operatorname{curl}\mathbf{F} = \mathbf{0}$

8. (7 points) Let E be the portion of the ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant: $x \geq 0$, $y \geq 0$, $z \geq 0$.

   Rewrite the triple integral below as an iterated integral in spherical coordinates and evaluate.

$$\iiint_E 5y\,dV$$

   Solution:

$$\begin{aligned} 5\iiint_E y\,dV &= 5\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\sin\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 5\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^1\rho^3\,d\rho \\ &= 5\times 1\times\frac{\pi}{4}\times\frac{1}{4} \end{aligned}$$

9. (7 points) Let $\mathbf{F}(x,y) = \langle y^2, x^2\rangle$. Use Green's Theorem to calculate the work done by $\mathbf{F}$ on a particle moving counter-clockwise around the triangle bounded by $x = 0,\ y = 0,\ x + y = 1$.

   Solution:

   Let T be the given triangle and let R be its interior. Then

$$\begin{aligned} \oint_T \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial\left(x^2\right)}{\partial x} - \frac{\partial\left(y^2\right)}{\partial y}\right) dA \\ &= \int_0^1\int_0^{1-x}(2x - 2y)\,dy\,dx \\ &= \int_0^1 (2xy - y^2)\Big|_{y=0}^{y=1-x}\,dx \\ &= \int_0^1 (4x - 3x^2 - 1)\,dx \\ &= (2x^2 - x^3 - x)\Big|_0^1 = 0 \end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

10. (4 points) Parameterize the part of the plane $x + z = 3$ that lies above the disk $(x-1)^2 + y^2 \leq 1$.

A. $\mathbf{r}(s,t) = \langle s, t, 3-s \rangle$ with $s \in [0,2]$ and $t \in [-1,1]$.

B. $\mathbf{r}(s,t) = \langle s, t, 3-s \rangle$, with $s \in [0, 2\cos t]$, and $t \in [-1,1]$.

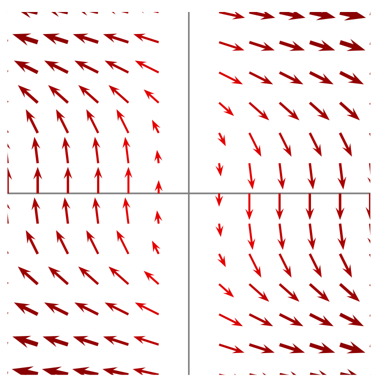
C. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 3 - s\cos t \rangle$ **with** $s \in [0, 2\cos t]$ **and** $t \in [-\pi/2, \pi/2]$**.**

D. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 3 - s\cos t \rangle$ with $s \in [0,2]$ and $t \in [-\pi/2, \pi/2]$.

E. None of the above

11. (4 points) Which of the following vector field plots could be $\mathbf{F} = xy^2\,\mathbf{i} - x\,\mathbf{j}$?

C.



12. (4 points) Let $\mathbf{F} = \langle x^2 y, -yz, z^2 \rangle$. Which of the following is true?

A. $\text{curl}\,\mathbf{F} = y - x^2$ and $\text{div}\,\mathbf{F} = 2xy + z$

B. $\text{curl}\,\mathbf{F} = \langle 2xy, z, 0 \rangle$ and $\text{div}\,\mathbf{F} = \langle y, 0, -x^2 \rangle$

C. $\text{curl}\,\mathbf{F} = 2xy + z$ and $\text{div}\,\mathbf{F} = y - x^2$

D. **curl** $\mathbf{F} = \langle y, 0, -x^2 \rangle$ **and div** $\mathbf{F} = 2xy + z$

E. None of the above

13. (4 points) Let $P = P(1/2, 2)$ and $\mathbf{u} = \frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}$. If $\nabla f(P) = 3\,\mathbf{i} + 2\,\mathbf{j}$, then $D_{\mathbf{u}}(P) =$

A. $\frac{11}{2}$

B. $\frac{3}{\sqrt{2}}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{-3}{2\sqrt{2}}$

E. None of the above

14. (4 points) Suppose that the line segment C is given by the parametrization: $\mathbf{r}(t) = (t+1)\,\mathbf{i} + 2t\,\mathbf{j},\ 0 \le t \le 3$. Evaluate the integral below.

$$\int_C \langle x^2, -y \rangle \cdot d\mathbf{r}$$

A. 21

B. 57

C. 39

D. 3

E. None of the above

15. (4 points)

$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\, dy\, dz\, dx =$$

A. $\frac{-\cos 8}{2}$

B. $\frac{-\cos 8}{4}$

C. $\frac{\sin 8}{4}$

D. $\frac{-\cos 4}{2}$

E. None of the above

16. (4 points) Identify the surface given by the vector equations $\mathbf{r}(u,v) = \langle u, 5\sin 3v, 4\cos 3v \rangle$.

A. plane

B. elliptic paraboloid

C. cylinder

D. ellipsoid

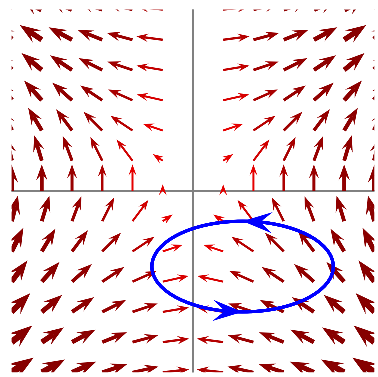
E. None of the above

17. (4 points) Let f be a differentiable function of x and y with continuous second order partial derivatives. Let $\mathbf{F} = \nabla f = M\,\mathbf{i} + N\,\mathbf{j}$ be a gradient field and let C be the (positively oriented) ellipse as shown in the sketch below. Consider the following statements.

(a) $\mathbf{F}$ is a conservative vector field.

(b) $\oint_C M\,dx + N\,dy = 0$

(c) $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k} > 0$



A. All three statements are true.

B. Only (a) and (b) are true.

C. Only (a) and (c) are true.

D. Only (b) and (c) are true.

E. None of the above

18. (4 points) Let $\mathbf{F}(x,y) = \nabla \arctan\left(\dfrac{y}{x}\right) = \dfrac{-y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$.

If C is the unit circle parameterized by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} =$

A. 0

B. 2π

C. π

D. Undefined

E. None of the above