Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) Find a parametrization of the line of intersection between planes $3x + y - 2z = 7$ and $2x - 2y + 3z = 1$.

Solution: Multiplying the first equation by 2 and adding it to the second yields.

$$8x - z = 15$$

One paramtrization would then be

$$x = t, \quad y = 13t - 23, \quad z = 8t - 15$$

Of course, there are others.

2. (7 points) What is the length of the curve $\mathbf{r}(t) = \left\langle 12t, 8t^{3/2}, 3t^2 \right\rangle$ from $t = 0$ to $t = 1$?

Solution:

$$\frac{d\mathbf{r}}{dt} = 12\,\mathbf{i} + 12\sqrt{t}\,\mathbf{j} + 6t\,\mathbf{k}$$

$$\left|\frac{d\mathbf{r}}{dt}\right|^2 = 144 + 144t + 36t^2 = 36(2+t)^2$$

It follows that

$$\begin{aligned} L &= \int_0^1 6|2+t|\,dt = 6\int_0^1 (2+t)\,dt \\ &= 6\left(2t + \frac{t^2}{2}\right)\Big|_0^1 \\ &= 15 \end{aligned}$$

3. (7 points) Evaluate the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} &= \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}\,\frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2} \\
&= \lim_{(x,y)\to(2,0)} \frac{2x-y-4}{2x-y-4}\,\frac{1}{\sqrt{2x-y}+2} \\
&= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2} \\
&= \frac{1}{4}
\end{aligned}$$

4. (7 points) Evaluate the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4}{x^4+4y^2}$

Solution: We claim the limit does not exist. To see this we evaluate the limit along the curve $y = cx^2,\ c \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{3x^4}{x^4+4y^2} &= \lim_{(x,cx^2)\to(0,0)} \frac{3x^4}{x^4+4c^2x^4} \\
&= \lim_{(x,cx^2)\to(0,0)} \frac{3x^4}{x^4(1+4c^2)} \\
&= \lim_{(x,cx^2)\to(0,0)} \frac{3}{1+4c^2}
\end{aligned}$$

Since the limit varies with c, the result follows. *Note:* Other choices are possible. For example, one can evaluate limits along the coordinate axes.

5. Let $f(x, y) = x^2 \cos y - 3y\sqrt{x}$ and answer the following questions

(a) (5 points) Calculate the partial derivatives, f_x and f_y.

Solution:

$$f_x = 2x \cos y - \frac{3y}{2\sqrt{x}}$$
$$f_y = -x^2 \sin y - 3\sqrt{x}$$

(b) (5 points) Use your answer in part (a) to find the linearization of f at $(4, 0)$.

Solution:

$$f(4,0) = 16, \quad f_x(4,0) = 8, \quad f_y(4,0) = -6$$

It follows that

$$\begin{aligned} L(x,y) &= f(4,0) + f_x(4,0)(x-4) + f_y(4,0)(y-0) \\ &= 16 + 8(x-4) - 6y \end{aligned}$$

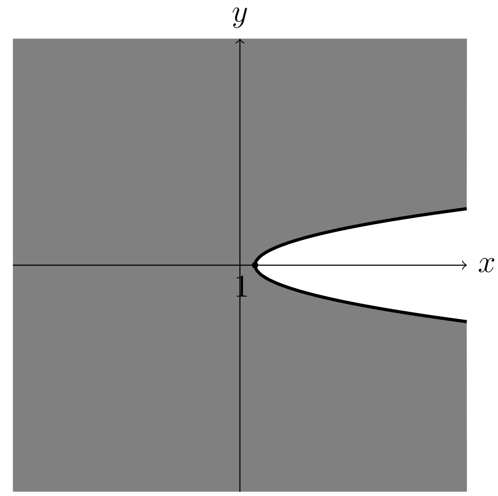
(c) (4 points) Use your answer in part (b) to approximate $f(3.9, 0.15)$.

Solution:

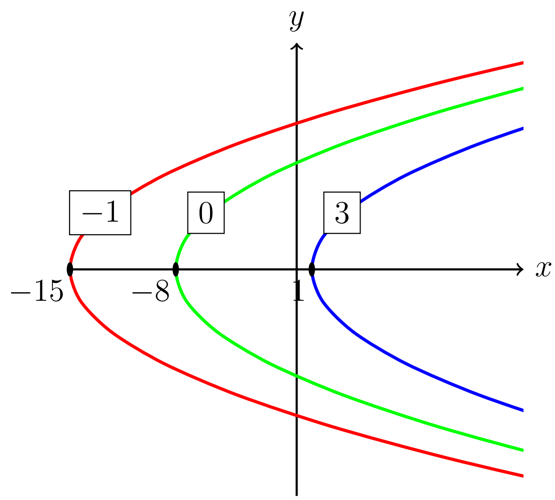
$$\begin{aligned} f(3.9, 0.15) &\approx L(3.9, 0.15) \\ &= 16 + 8(3.9 - 4) - 6(0.15) \\ &= 14.3 \end{aligned}$$

6. Consider the function $f(x, y) = 3 - \sqrt{y^2 - x + 1}$

(a) (5 points) Sketch the domain of f.



(b) (6 points) Sketch the level curves of f for levels $k = -1$, $k = 0$, and $k = 3$.



(c) (3 points) What is the range of f?

Solution: $(-\infty, 3]$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) The intersection of the quadric surface $z = 3x^2 - 4y^2$ and the plane $z = -2$ is:

   A. one point

   B. two straight lines

   C. an ellipse

   D. a parabola

   E. a hyperbola

8. (4 points) Find the equation of the tangent plane to the surface $z = 2x^2 - y^2 + 3y + 1$ at the point $(1, 2, 5)$.

   A. $z = 3 + 4x - 2y$

   B. $z = 5 + 4(x - 1) - (y - 2)$

   C. $z = 5 + 4(x - 1) - 2(y - 2)$

   D. $z = 3 + 2(x - 1) - (y - 2)$

   E. None of the above.

9. (4 points) The contour plot (level curves) to the right could be from which function?

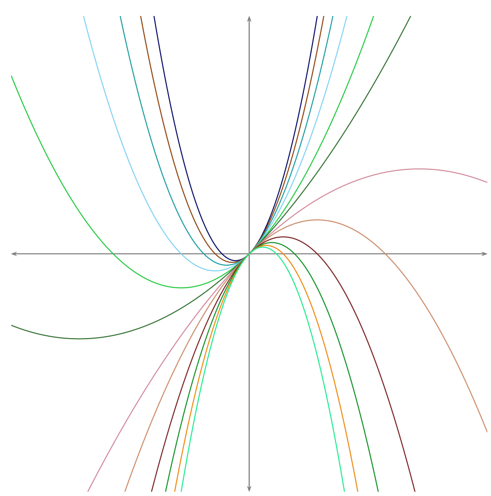
   A. $f(x, y) = x^2 - y$

   B. $f(x, y) = \dfrac{y^2}{x - y}$

   C. $f(x, y) = \dfrac{x - y}{x^2}$

   D. $f(x, y) = \dfrac{x^2 - y}{x^2}$

   E. $f(x, y) = \sqrt{y - x^2}$



10. (4 points) Find the area of the triangle with vertices $A(1,0,0)$, $B(1,0,2)$ and $C(2,3,4)$

A. $\sqrt{5}/2$

B. $\sqrt{40}$

C. $\sqrt{10}$

D. $\sqrt{10}/2$

E. $\sqrt{20}$

11. (4 points) Let $f(x,y) = x^2\sqrt{y} + xy$. Evaluate $f_{xx}f_{yy} - f_{xy}^2$ at $P(2,1)$. *Hint:* $f_{yy}(P) = -1$.

A. -11

B. -6

C. -7

D. 2

E. 9

12. (4 points) Which of the following is the unit tangent vector to the curve at $t = 0$ if $\mathbf{r}(t) = \langle 2\cos t, -2\sin t, \sqrt{5}\,t \rangle$.

A. $\mathbf{T}(0) = \langle 1, 0, 0 \rangle$

B. $\mathbf{T}(0) = \left\langle \frac{2}{13}, \frac{-2}{13}, \frac{\sqrt{5}}{13} \right\rangle$

C. $\mathbf{T}(0) = \left\langle \frac{2}{3}, 0, \frac{\sqrt{5}}{3} \right\rangle$

D. $\mathbf{T}(0) = \langle 2, 2, \sqrt{5} \rangle$

E. $\mathbf{T}(0) = \left\langle 0, \frac{-2}{3}, \frac{\sqrt{5}}{3} \right\rangle$

13. (4 points) Let $\mathbf{A} = \langle 1, -2, 3\rangle$ and $\mathbf{B} = \langle 2, 1, -2\rangle$ and let θ be the angle between $\mathbf{A}$ and $\mathbf{B}$. Then $\cos\theta =$

A. $\frac{1}{\sqrt{14}}$

B. $\frac{-1}{\sqrt{14}}$

C. $\frac{2}{\sqrt{14}}$

D. $\frac{-2}{\sqrt{14}}$

E. $\frac{-6}{14}$

14. (4 points) Suppose $z = x + 3xy^2$, where x and y are differentiable functions of t with

$$x(0) = 3, \quad x'(0) = 2, \quad y(0) = 1, \quad y'(0) = 1/2.$$

Then $\left.\frac{dz}{dt}\right|_{t=0} = 17$

A. 13/2

B. 24

C. 22

D. 9

E. None of the above.

15. (4 points) Find a unit vector orthogonal to the plane $3(x-1) - 2(y-4) + z - 4 = 0$.

A. $\left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$

B. $\left\langle \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}} \right\rangle$

C. $\left\langle \frac{1}{\sqrt{33}}, \frac{4}{\sqrt{33}}, \frac{4}{\sqrt{33}} \right\rangle$

D. $\langle 1, 4, 4\rangle$

E. None of the above.

More Challenging Problem(s). Show all work to receive credit.

16. (7 points) Find the projection of the vector $\mathbf{u} = 3\,\mathbf{i} + 2\,\mathbf{j} - \,\mathbf{k}$ onto the line passing through the points $P(1, 2, -3)$ and $Q(2, 3, 3)$.

Solution: Let $\mathbf{v} = \overline{PQ} = \,\mathbf{i} + \,\mathbf{j} + 6\,\mathbf{k}$. Then

$$\begin{aligned} \text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} = \frac{3+2-6}{\sqrt{14}\sqrt{38}}\,(\,\mathbf{i} + \,\mathbf{j} + 6\,\mathbf{k}) \\ &= \frac{-1}{38}\,(\,\mathbf{i} + \,\mathbf{j} + 6\,\mathbf{k}) \end{aligned}$$

17. (7 points) Find the position and velocity functions of a particle that satisfies the following conditions.

$$\begin{aligned} \mathbf{a}(t) &= 2\,\mathbf{i} + \frac{3}{\sqrt{1+t}}\,\mathbf{j} + 9e^{3t}\,\mathbf{k} \\ \mathbf{v}(0) &= -\,\mathbf{i} + 6\,\mathbf{j} + 3\,\mathbf{k} \\ \mathbf{r}(0) &= 4\,\mathbf{j} \end{aligned}$$

Solution: Integrating the first equation yields

$$\begin{aligned} \mathbf{v}(t) &= 2t\,\mathbf{i} + 6\sqrt{1+t}\,\mathbf{j} + 3e^{3t}\,\mathbf{k} + \mathbf{C}_1 \\ &= (2t-1)\,\mathbf{i} + 6\sqrt{1+t}\,\mathbf{j} + 3e^{3t}\,\mathbf{k} \end{aligned}$$

after applying the initial conditions. Integrating again yields

$$\mathbf{r}(t) = (t^2 - t)\,\mathbf{i} + 4(1+t)^{3/2}\,\mathbf{j} + e^{3t}\,\mathbf{k} + \mathbf{C}_2$$

The initial conditions imply that $\mathbf{C}_2 = -\,\mathbf{k}$. This yields

$$\mathbf{r}(t) = t^2 - t)\,\mathbf{i} + 4(1+t)^{3/2}\,\mathbf{j} + (e^{3t} - 1)\,\mathbf{k}$$

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\mathbf{u}||\mathbf{v}| \cos\theta$$
- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\, \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Trigonometry

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$