Name: ____________________

Section: __________ **Recitation Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (9 points) Let $g(x, y) = x^2y - 4xy - 6y^2$. Find and classify each critical point of g as a local minimum, a local maximum, or a saddle point.

Solution:

$$g_x = 2xy - 4y$$
$$g_y = x^2 - 4x - 12y$$

So the critical points are $P(2, -1/3)$, $O(0, 0)$, and $Q(4, 0)$. Now let

$$\begin{aligned} D &= g_{xx}g_{yy} - g_{xy}^2 \\ &= (2y)(-12) - (2x-4)^2 \end{aligned}$$
$$D(O) = -16 < 0 \implies g \text{ has a saddle point at } O$$
$$D(Q) = -16 < 0 \implies g \text{ has a saddle point at } Q$$
$$D(P) = 8 > 0 \text{ and } g_{xx}(P) = -2/3 \implies g \text{ has a local maximum at } P$$

2. (5 points) Let E be the hemisphere $x^2 + y^2 + z^2 \leq 9$ with $z \geq 0$. Express that triple integral below in spherical coordinates. **Do Not Evaluate.**

$$\iiint_E e^{x^2+y^2+z^2}\, dV$$

Solution:

$$\iiint_E e^{x^2+y^2+z^2}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 e^{\rho^2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

3. Let $\mathbf{F} = (3yz - 2e^x \sin z)\,\mathbf{i} + (3xz + 2y)\,\mathbf{j} + (3xy - 2e^x \cos z)\,\mathbf{k}$ and answer the questions below.

(a) (8 points) Find a function f so that $\nabla f = \mathbf{F}$.

Solution:

$$f = 3xyz - 2e^x \sin(z) + y^2$$

(b) (3 points) Let C be the line segment from $P(0, 1, \pi/2)$ to $Q(1/2, 2, \pi)$. Evaluate the integral

$$\int_C (3yz - 2e^x \sin z)\, dx + (3xz + 2y)\, dy + (3xy - 2e^x \cos z)\ dz$$

Solution:

$$= f(Q) - f(P) = 3\pi + 4 - (-1)$$

(c) (3 points) Let P and Q be as defined above and let $R = R(1, 3, 2\pi)$. Now let C_1 be the line segment from P to R and let C_2 be the line segment from R to Q. Evaluate the integral

$$\int_{C_1 \cup C_2} (3yz - 2e^x \sin z)\, dx + (3xz + 2y)\, dy + (3xy - 2e^x \cos z)\ dz$$

Solution: Same as part (b).

4. (7 points) A solid D lies within the cylinder $x^2 + y^2 = 4$ and is bounded below by the plane $z = 0$ and above by the plane $z = 6$. The temperature at each point in D is given by $T(x, y, z) = x + y + z + 1$. Find the average temperature over the solid D.

Solution: The volume of the solid D is 24π. It follows that the average temperature over D is given by

$$\begin{aligned}
T_{\text{avg}} &= \frac{1}{24\pi}\iiint_D T(x,y,z)\,dV \\
&= \frac{1}{24\pi}\int_0^2\int_0^6\int_0^{2\pi} r^2\cos\theta + r^2\sin\theta\,d\theta\,dz\,dr + \frac{1}{24\pi}\int_0^{2\pi}\int_0^2\int_0^6 (z+1)r\,dz\,dr\,d\theta \\
&= 0 + \frac{4\pi}{24\pi}\int_0^6 z + 1\,dz \\
&= \frac{1}{6}\times 24
\end{aligned}$$

5. (7 points) Let $\mathbf{F} = \langle x + 3y, 2x - y\rangle$. Use Green's Theorem to calculate the work done by $\mathbf{F}$ on a particle moving counter-clockwise around the triangle with vertices $(0, 0), (2, 0),$ and $(2, 5)$.

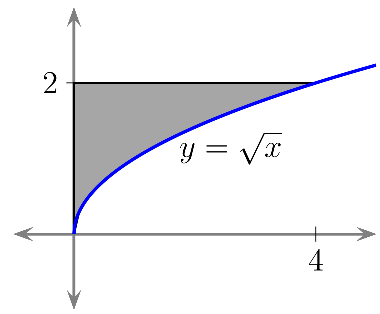
Solution: Let T be the given triangle and R be its interior. Then

$$\begin{aligned}
\oint_T \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial(2x-y)}{\partial x} - \frac{\partial(x+3y)}{\partial y}\right) dA \\
&= \iint_R (2-3)\,dy\,dx \\
&= -1 \times \text{Area of R} \\
&= -5
\end{aligned}$$

6. Consider the integral below and answer the questions that follow.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{\sqrt{1+y^3}}\, dy\, dx$$

(a) (3 points) Sketch the region of integration.
Label any relevant intersection points.



(b) (5 points) Evaluate the integral above by reversing the order of integration.

Solution:

$$\begin{aligned}\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{\sqrt{1+y^3}}\, dy\, dx &= \int_0^2 \frac{1}{\sqrt{1+y^3}} \int_0^{y^2} dx\, dy \\ &= \int_0^2 \frac{y^2}{\sqrt{1+y^3}}\, dy \\ &= \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\, du \\ &= \frac{2}{3}(\sqrt{9}-1) = \frac{4}{3}\end{aligned}$$

7. (6 points) Find the surface area of the portion of the paraboloid $z = 2 - x^2 - y^2$ that lies above the plane $z = 0$.

Solution: Let R be the disk of radius $\sqrt{2}$ centered at the origin. Then

$$\begin{aligned}\text{Area} &= \iint_R \sqrt{1+(2x)^2+(2y)^2}\, dA \\ &= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr\, d\theta \\ &= 2\pi \int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr \\ &= \frac{2\pi}{8}\int_1^9 \sqrt{u}\, du \\ &= \frac{\pi}{6}(9^{3/2}-1) = \frac{13\pi}{3}\end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Let $\mathbf{F} = \langle x^2, xyz, yz \rangle$. Which of the following is true?

A. $\operatorname{curl}\mathbf{F} = 2x - y$ and $\operatorname{div}\mathbf{F} = yz - xy + z$

B. $\operatorname{curl}\mathbf{F} = \langle 2x, xz, y \rangle$ and $\operatorname{div}\mathbf{F} = \langle z, -xy, yz \rangle$

C. $\operatorname{curl}\mathbf{F} = \langle 2x, xz, y \rangle$ and $\operatorname{div}\mathbf{F} = z - xy$

D. $\operatorname{curl}\mathbf{F} = \langle z - xy, 0, yz \rangle$ and $\operatorname{div}\mathbf{F} = 2x + xz + y$

E. None of the above.

9. (4 points) Let $P = P(-1, 5)$ and $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$. If $\nabla f(P) = 2\mathbf{i} + 4\mathbf{j}$, then $D_{\mathbf{u}}(P) =$

A. -7

B. $2\mathbf{i} - 12\mathbf{j}$

C. $-\sqrt{10}$

D. $-\sqrt{5}$

E. None of the above.

10. (4 points) $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos x^2}{2\sqrt{z}}\, dx\, dy\, dz =$

A. $8\sin 4$

B. $4\sin 2$

C. $2\sin 4$

D. $\sqrt{8}\sin 2$

E. $\sin 16$

11. (4 points) The surface $\phi = \dfrac{4\pi}{5}$ can be described as a

A. Cylinder

B. Plane

C. Paraboloid

D. Half-Cone

E. None of the above.

12. (4 points) Let R be the set of points (x, y) that satisfy $4x^2 + y^2 \leq 4$. Then the absolute maximum value of the function $f(x, y) = 4x^2 + 5y$ on R is

A. f has no maximum value on R.

B. $41/4$

C. 10

D. 16

E. None of the above.

13. (4 points) Let C be the curve whose parametrization is given by the vector equation $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $0 \leq t \leq 1$. Then $\displaystyle\int_C 6x\,ds =$

A. 3

B. $\frac{1}{2}\left(5^{3/2} - 1\right)$

C. $\frac{1}{2}\left(2^{3/2} - 1\right)$

D. $\frac{1}{2}\left(\sqrt{5} - 1\right)$

E. None of the above.

14. (4 points) At the point $P(1,2,3)$, in which direction does the function $f(x,y,z) = xy - z^2 + \dfrac{y}{x}$ increase most rapidly?

A. $\langle 0, 2, -6 \rangle$

B. $\langle 0, -2, 6 \rangle$

C. $\langle 1, 2, 3 \rangle$

D. $\langle 2, 0, -6 \rangle$

E. $\langle -2, 0, 6 \rangle$

15. (4 points) Find the work done by the force $\mathbf{F} = \langle -y, 4x \rangle$ along the straight-line segment from $(1,2)$ to $(3,3)$.

A. $11/2$

B. 6

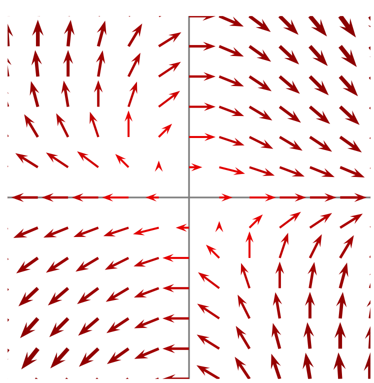
C. $31/2$

D. 13

E. 3

16. (4 points) Which of the following vector field plots could be $\mathbf{F} = (x+y)\,\mathbf{i} - xy\,\mathbf{j}$?

D.



More Challenging Problem(s). Show all work to receive credit.

17. (4 points) Let $f(x,y) = x^2 - 2xy^2 + y^4$ and observe that f has a critical point at $P(0,0)$. Consider the following statements.

(a) f has a local minimum at P.

(b) The second derivative test is inconclusive at P.

(c) f has a local maximum at P.

A. Only (a) is true.

B. Only (b) is true.

C. Only (c) is true.

D. Only (a) and (b) are true.

E. Only (b) and (c) are true.

18. (10 points) Let R be the region in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$ (see Fig. 1). Evaluate the integral below in polar coordinates. No credit for any other method.

Solution: Notice that the region R intersects the circle at $\pi/6 \le \theta \le \pi/3$ and radial lines from the origin enter the region at $r = \csc\theta$ and exit at $r = 2$.

$$\iint_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} (4 - \csc^2\theta)\,d\theta$$

$$= \frac{1}{2}(4\theta + \cot\theta)\Big|_{\pi/6}^{\pi/3} = \frac{\pi - \sqrt{3}}{3}$$

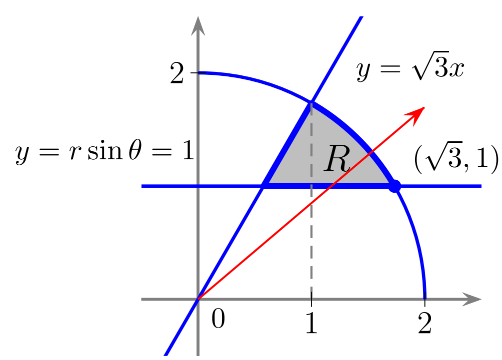


Figure 1: Region R

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are $\boxed{\text{BOXED}}$.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle \\ \mathbf{j} &= \langle 0, 1, 0 \rangle \\ \mathbf{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= |\mathbf{u}||\mathbf{v}| \cos\theta \end{aligned}$$
- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\operatorname{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\, \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u}$
- **Second Derivative Test** Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
  - (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - (c) If $D < 0$ then $f(a, b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

Multiple Integrals

- **Area**: $A(D) = \iint_D 1 \; dA$
- **Volume**: $V(E) = \iiint_E 1 \; dV$

Polar/Cylindrical

- Transformations

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
x &= r\cos\theta \\
y &= r\sin\theta \\
y/x &= \tan\theta
\end{aligned}
$$

- $\iint_D f(x,y)\; dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$
- $\iiint_E f(x,y,z)\; dV =$
  $\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$

Spherical

- Transformations

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi \\
\rho^2 &= x^2 + y^2 + z^2
\end{aligned}
$$

- $\iiint_E f(x,y,z)\; dV =$

$\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- $\mathbf{F}$ is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem**

$$\int_C P\; dx + Q\; dy = \iint_D (Q_x - P_y)\, dA$$