Name: ____________________

Section: ____________ **Recitation Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (5 points) Find a parametrization of the line of intersection between planes $6x + 2y - 3z = 8$ and $4x - y + 2z = 4$.

Solution: Multiplying the second equation by 2 and adding it to the first yields

$$14x + z = 16$$

So let $x = t$. Then

$$\begin{aligned} z &= 16 - 14x = 16 - 14t \\ y &= 4x + 2z - 4 = 4t + 2(16 - 14t) - 4 = 28 - 24t \end{aligned}$$

2. (9 points) Let $g(x, y) = 6y^2 - 4y^3 - 3x^2 + 6xy$. Find and classify each critical point of g as a local minimum, a local maximum, or a saddle point. *Be sure to justify your response.*

Solution:

$$g_x = 6y - 6x \text{ and } g_y = 12y - 12y^2 + 6x$$

Together these equations imply that g has critical points at $P(0, 0)$ and $Q(3/2, 3/2)$. Now

$$g_{xx} = -6,\ g_{yy} = 12 - 24y \ \text{ and } g_{xy} = 6$$

It follows that

$$D(x, y) = g_{xx}g_{yy} - g_{xy}^2 = -6(12 - 24y) - 6^2 = 144y - 108$$

Now

$$\begin{aligned} &D(P) = -108 < 0 \\ &D(Q) = 108 > 0 \quad \text{and} \quad g_{xx}(Q) = -6 < 0 \end{aligned}$$

It follows by the second derivative test that g has a saddle point at $P(0, 0)$ and a local maximum at $Q(3/2, 3/2)$.

3. (7 points) Evaluate the limit below or show that it does not exist.

$$\lim_{(x,y)\to(0,1)} \frac{\sqrt{2x+1}-y}{y^2-2x-1}$$

Solution:

$$\begin{aligned}\lim_{(x,y)\to(0,1)} \frac{\sqrt{2x+1}-y}{y^2-2x-1} &= -\lim_{(x,y)\to(0,1)} \frac{\sqrt{2x+1}-y}{(\sqrt{2x+1}-y)(\sqrt{2x+1}+y)}\\ &= -\lim_{(x,y)\to(0,1)} \frac{1}{\sqrt{2x+1}+y}\\ &= \frac{-1}{2}\end{aligned}$$

4. (7 points) Find the equation of the plane tangent to the surface $z = 12 - x^2 - 3y^2$ at the point $P(2,-1,5)$. Also, find a vector equation of the line normal to the surface at P.

Solution:

$$\begin{aligned} z_x &= -2x \quad \Longrightarrow \quad z_x(P) = -4\\ z_y &= -6y \quad \Longrightarrow \quad z_y(P) = 6\end{aligned}$$

It follows that the tangent plane is given by the equation

$$z = 5 - 4(x-2) + 6(y+1)$$

It is easy to see that the vector normal to the tangent plane is any nonzero multiple of

$$\mathbf{n} = 4\,\mathbf{i} - 6\,\mathbf{j} + \mathbf{k}$$

It follows that the parametric equations of the line normal to the surface are

$$x = 2 + 4t, \quad y = -1 - 6t, \quad z = 5 + t$$

Tangent Plane: $z = 5 - 4(x-2) + 6(y+1)$

Normal Line: $\mathbf{r}(t) = \langle 2, -1, 5\rangle + t\,\langle 4, -6, 1\rangle$

5. Let $f(x,y) = 5y^3\,e^{x-2} - 2xy$ and answer the following questions.

(a) (4 points) Calculate the partial derivatives, f_x and f_y.

Solution:

$$f_x = 5y^3\,e^{x-2} - 2y \quad \text{and} \quad f_y = 15y^2\,e^{x-2} - 2x$$

(b) (4 points) Use your answer in part (a) to find the linearization of f at $(2,1)$.

Solution: Let $P = P(2,1)$. Then

$$f_x(P) = 3 \quad \text{and} \quad f_y(P) = 11$$

It follows that

$$\begin{aligned} L(x,y) &= f(P) + f_x(P)(x-2) + f_y(P)(y-1) \\ &= 1 + 3(x-2) + 11(y-1) \end{aligned}$$

(c) (6 points) Calculate the second partial derivatives, f_{xx}, f_{xy}, and f_{yy}.

Solution:

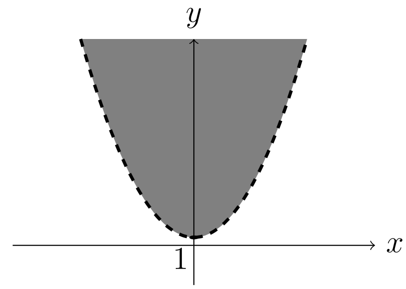
$$\begin{aligned} f_{xx} &= 5y^3e^{x-2} \\ f_{yy} &= 30ye^{x-2} \\ f_{xy} &= 15y^2e^{x-2} - 2 \end{aligned}$$

6. Consider the function $f(x,y) = \dfrac{1}{\sqrt{y - x^2 - 1}}$

(a) (5 points) Sketch the domain of f.

Solution:

$$y - x^2 - 1 > 0 \quad \Longrightarrow \quad y > x^2 + 1$$



(b) (2 points) What is the range of f? *Express your answer using interval notation.*

Solution: $(0, \infty)$

7. (7 points) Find the position and velocity functions of a particle that satisfies the following conditions.

$$\mathbf{a}(t) = -4\sin 4t\,\mathbf{i} + 6\,\mathbf{j} + 12e^{2t}\,\mathbf{k}$$
$$\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + 6\,\mathbf{k}$$
$$\mathbf{r}(0) = 3\,\mathbf{k}$$

$\mathbf{v}(t) =$ $\cos 4t\,\mathbf{i} + (6t - 1)\,\mathbf{j} + 6e^{2t}\,\mathbf{k}$

$\mathbf{r}(t) =$ $\dfrac{\sin 4t}{4}\,\mathbf{i} + (3t^2 - t)\,\mathbf{j} + 3e^{2t}\,\mathbf{k}$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Let θ be the angle between the vectors $\mathbf{A} = \langle 1, 2, 4\rangle$ and $\mathbf{B} = \langle 1, -1, 1\rangle$. Then $\cos\theta =$

A. $\dfrac{3}{\sqrt{21}}$

B. $\dfrac{1}{\sqrt{7}}$

C. $\dfrac{3}{7}$

D. $\dfrac{3}{\sqrt{7}}$

E. None of the above.

9. (4 points) At the point $P(-1, 1, 3)$, in which direction does the function $f(x, y, z) = xy - z^2 + \dfrac{4x}{y}$ **decrease** most rapidly?

A. $\langle 2, -1, 5\rangle$

B. $\langle -2, 1, -5\rangle$

C. $\langle -5, -3, 6\rangle$

D. $\langle 5, 3, -6\rangle$

E. None of the above.

10. (4 points) Let $P = P(3, 4)$ and $\mathbf{u} = 2\,\mathbf{i} - \mathbf{j}$. If $\nabla f(P) = -\,\mathbf{i} + 6\,\mathbf{j}$, then $D_{\mathbf{u}}(P) =$

A. $\dfrac{-8}{5}$

B. $2\,\mathbf{i} - 10\,\mathbf{j}$

C. $\dfrac{-8}{\sqrt{37}}$

D. $\dfrac{-8}{\sqrt{5}}$

E. None of the above.

11. (4 points) The intersection of the quadric surface $z = 3x^2 - 4y^2$ and the plane $y = 2$ is:

A. an ellipse

B. one line

C. a parabola

D. two lines

E. a hyperbola

12. (4 points) Suppose $z = 3xy^2 - \ln x$, where x and y are differentiable functions of t with

$$x(0) = 1, \quad x'(0) = 3, \quad y(0) = 1, \quad y'(0) = 1/2.$$

Then $\left.\dfrac{dz}{dt}\right|_{t=0} =$

A. Does not exist.

B. 8

C. 9

D. 10

E. None of the above.

13. (4 points) Find a unit vector orthogonal to the plane $5x - 2(y-1) + 3(z-4) = 0$.

A. $\dfrac{1}{\sqrt{38}}\langle 5, -2, -3\rangle$

B. $\dfrac{1}{\sqrt{38}}\langle -5, 2, -3\rangle$

C. $\dfrac{1}{\sqrt{17}}\langle 0, 1, -4\rangle$

D. $\dfrac{1}{\sqrt{17}}\langle 0, 1, 4\rangle$

E. None of the above.

14. (4 points) Find the area of the triangle with vertices $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,3)$.

A. $\dfrac{7}{2}$

B. $\dfrac{49}{2}$

C. $\sqrt{14}$

D. $\dfrac{\sqrt{14}}{2}$

E. None of the above.

15. (4 points) Let $z = \sin(\pi x)\, e^{5y}$. Then the differential $dz =$

A. $5\pi\cos(\pi x)\, e^{5y}\, dx\, dy$

B. $5\sin(\pi x)e^{5y}\, dx + \pi\cos(\pi x)\, e^{5y}\, dy$

C. $\pi\cos(\pi x)\, dx + 5e^{5y}\, dy$

D. $\pi\cos(\pi x)\, e^{5y}\, dx + 5\sin(\pi x)e^{5y}\, dy$

E. None of the above.

16. (4 points) Let $\mathbf{r}(t) = \langle 2e^{3t}, e^{3t}, 2t\rangle$. Which of the following is the unit tangent vector to the curve at $t = 0$?

A. $\mathbf{T}(0) = \dfrac{1}{3}\langle 2, 1, 2\rangle$

B. $\mathbf{T}(0) = \dfrac{1}{\sqrt{5}}\langle 2, 1, 0\rangle$

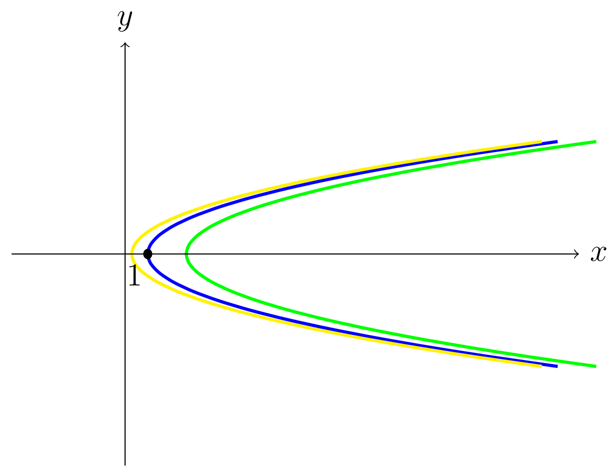
C. $\mathbf{T}(0) = \dfrac{1}{\sqrt{5e^2+4}}\langle 2e, e, 2\rangle$

D. $\mathbf{T}(0) = \dfrac{1}{7}\langle 6, 3, 2\rangle$

E. None of the above.

More Challenging Problem(s). Show all work to receive credit.

17. (7 points) Sketch the level curves of $g(x, y) = \ln(x - y^2)$ for levels $k = -1$, $k = 0$, and $k = 1$.



18. (7 points) Evaluate the limit below or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

Solution: We claim the limit does not exist. To see this, let $y = mx$, for some arbitrary $m \in \mathbb{R}$ (other choices should also work). Then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x\to 0} \frac{mx^2}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

So the limit value will vary as a function of m. We conclude that the limit does not exist.

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are $\boxed{\text{BOXED}}$.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1,0,0\rangle$$
$$\mathbf{j} = \langle 0,1,0\rangle$$
$$\mathbf{k} = \langle 0,0,1\rangle$$

- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$

- **Dot Product**:
$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= |\mathbf{u}||\mathbf{v}|\cos\theta\end{aligned}$$

- **Cross Product**:
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

- **Vector Projection**: $\quad \text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\quad \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$

- **Plane** normal to $\mathbf{n} = \langle a,b,c\rangle$:
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a,b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\,dt$$

- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Surfaces

- **Directional Derivative:** $D_{\mathbf{u}}f(x,y) = \nabla f\cdot\mathbf{u}$

- **Second Derivative Test** Suppose $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let
$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$
  - (a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
  - (b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
  - (c) If $D < 0$ then $f(a,b)$ is a saddle point.

Trigonometry

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x\cos x$