Name: ____________________

Section: ____________ **Recitation Instructor**: ____________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ____________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) Evaluate the iterated integral below.

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$$

Solution:

$$= \int_0^1 \int_x^{2x} xyz^2 \Big|_{z=0}^{z=y} dy\, dx = \int_0^1 \int_x^{2x} xy^3\, dy\, dx$$

$$= \frac{1}{4} \int_0^1 xy^4 \Big|_{y=x}^{y=2x} dx = \frac{15}{4} \int_0^1 x^5\, dx$$

$$= \frac{5}{8} x^6 \Big|_0^1 = \frac{5}{8}$$

2. (7 points) Let E be the solid bounded below by the **lower half** of the sphere $x^2 + y^2 + (z-1)^2 = 1$ and above by the cone $z = \sqrt{x^2 + y^2}$. Express the triple integral below as an iterated integral in spherical coordinates and evaluate.

$$\iiint_E dV$$

Solution:

$$\iiint_E dV = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi\, d\phi\, d\rho$$

$$= \frac{16\pi}{3} \int_{\pi/4}^{\pi/2} \cos^3\phi\, \sin\phi\, d\phi = \frac{4\pi}{3} \cos^4\phi \Big|_{\pi/2}^{\pi/4}$$

$$= \frac{4\pi}{3} \left[\left(\frac{\sqrt{2}}{2} \right)^4 \right] = \frac{\pi}{3}$$

3. Let $\mathbf{F} = (2x + 5y^2z)\ \mathbf{i} + (10xyz - 3e^{3y}\cos z)\ \mathbf{j} + (5xy^2 + e^{3y}\sin z)\ \mathbf{k}$ and answer the questions below.

(a) (8 points) Find a function f so that $\nabla f = \mathbf{F}$.

Solution:

$$f = 5xy^2z - e^{3y}\cos z + x^2$$

(b) (3 points) Let C be the union of the line segments from $P(2, 0, \pi)$ to $R(0, 0, 0)$ and from R to $Q(1, 1, \pi/2)$. Evaluate the integral

$$\int_C (2x + 5y^2z)\,dx + (10xyz - 3e^{3y}\cos z)\,dy + (5xy^2 + e^{3y}\sin z)\,dz = I$$

Solution: The vector field is conservative, so by the Fundamental Theorem of Line Integrals we have

$$I = f(Q) - f(P) = 1 + \frac{5\pi}{2} - 5$$

(c) (3 points) Find the work done by the force field $\mathbf{F}$ as a particle moves from P to Q along the path C as defined in part (b).

Solution: Since the vector field is conservative, the line integral is independent of path. So the answer is the same as part (b).

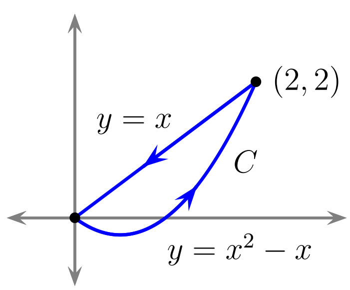
4. (7 points) A solid D lies within the cylinder $x^2 + y^2 = 9$ and is bounded below by the plane $z = 0$ and above by the plane $z = 2$. Let $f(x, y, z) = x + 3z + 1$. Evaluate the integral below.

$$\iiint_D f(x,y,z)\,dV$$

Solution:

$$\begin{aligned}\iiint_D f(x,y,z)\,dV &= \int_0^{2\pi}\int_0^3\int_0^2 r^2\cos\theta + 3rz + r\,dz\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^3 r^2\cos\theta\,dr\,d\theta + 4\pi\int_0^3 r\int_0^2 (3z+1)\,dz\,ddr \\ &= 0 + 4\pi\int_0^3 r\left(\frac{z^2}{2}+z\right)\Big|_{z=0}^{z=2} dr \\ &= 16\pi\int_0^3 r\,dr \\ &= 8\pi \times 9 = 72\pi\end{aligned}$$

5. (7 points) Let $\mathbf{F} = \langle -x^3y, x^4\rangle$. Use Green's Theorem to calculate the work done by $\mathbf{F}$ on a particle moving counterclockwise around curve C shown in the sketch below.



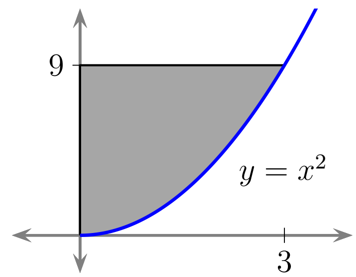
Solution: Let R be region bounded by curve C. Then by Green's Theorem

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial(x^4)}{\partial x} - \frac{\partial(-x^3y)}{\partial y}\right) dA \\ &= 5\iint_R x^3\,dy\,dx \\ &= 5\int_0^2\int_{x^2-x}^{x} x^3\,dy\,dx = 5\int_0^2 2x^4 - x^5\,dx \\ &= \frac{32}{3}\end{aligned}$$

6. Consider the integral below and answer the questions that follow.

$$\int_0^3 \int_{x^2}^9 \frac{\sqrt{1+\sqrt{y}}}{y}\, dy\, dx$$

(a) (3 points) Sketch the region of integration.
Label any relevant intersection points.



(b) (5 points) Evaluate the integral above by reversing the order of integration.

Solution:

$$\begin{aligned}\int_0^3 \int_{x^2}^9 \frac{\sqrt{1+\sqrt{y}}}{y}\, dy\, dx &= \int_0^9 \frac{\sqrt{1+\sqrt{y}}}{y} \int_0^{\sqrt{y}} dx\, dy \\ &= \int_0^9 \frac{\sqrt{1+\sqrt{y}}}{y} \sqrt{y}\, dy = \int_0^9 \frac{\sqrt{1+\sqrt{y}}}{\sqrt{y}}\, dy \\ &= 2\int_1^4 \sqrt{u}\, du, \qquad (u = 1+\sqrt{y},\ \text{etc.}) \\ &= 28/3\end{aligned}$$

7. (6 points) Let $\mathbf{F} = \langle x, 2y+1, z\rangle$. Find the work done by $\mathbf{F}$ over the curve C defined below in the direction of increasing t.

$$C:\ \mathbf{r}(t) = \left\langle \cos \pi t, t^2, \sin \pi t \right\rangle,\ 0 \le t \le 2$$

Solution:

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \cos \pi t\, \mathbf{i} + (2t^2+1)\, \mathbf{j} + \sin \pi t\, \mathbf{k} \\ d\mathbf{r} &= (-\pi \sin \pi t\, \mathbf{i} + 2t\, \mathbf{j} + \pi \cos \pi t\, \mathbf{k})\, dt\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (-\pi \sin \pi t\, \cos \pi t + 2t(2t^2+1) + \pi \sin \pi t\, \cos \pi t)\, dt \\ &= 2t(2t^2+1)\, dt\end{aligned}$$

so that

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (4t^3 + 2t)\, dt = 20$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Let $\mathbf{F} = \langle xz, xyz^2, yz \rangle$. Which of the following is true?

   A. $\operatorname{curl}\mathbf{F} = \langle xz, xyz^2, yz \rangle$ and $\operatorname{div}\mathbf{F} = 2xyz - yz$

   B. $\operatorname{curl}\mathbf{F} = \langle z - 2xyz, x, yz^2 \rangle$ and $\operatorname{div}\mathbf{F} = \langle y, z, xz^2 \rangle$

   C. $\operatorname{curl}\mathbf{F} = \left\langle z - 2xyz, x, yz^2 \right\rangle$ and $\operatorname{div}\mathbf{F} = y + z + xz^2$

   D. $\operatorname{curl}\mathbf{F} = \langle y, z, xz^2 \rangle$ and $\operatorname{div}\mathbf{F} = z - 2xyz + x + yz^2$

   E. None of the above.

9. (4 points) Let C be the curve whose parametrization is given by the vector equation $\mathbf{r}(t) = \langle 2t, t \rangle$, $0 \le t \le 1$. Then $\displaystyle\int_C (xy + y)\, ds =$

   A. $\sqrt{5}$

   B. $\dfrac{7\sqrt{5}}{6}$

   C. $\dfrac{3\sqrt{5}}{2}$

   D. $\dfrac{4\sqrt{5}}{2}$

   E. None of the above.

10. (4 points) $\displaystyle\int_0^2 \int_{2y}^4 \frac{2}{\sqrt{1+x^2}}\, dx\, dy =$

   A. $\dfrac{\ln 17}{2}$

   B. 4

   C. $\sqrt{17} - 1$

   D. $\dfrac{\sqrt{17} - 1}{2}$

   E. None of the above.

11. (4 points) The surface $\rho^2 - 2\rho \sin\phi \cos\theta = 8$ can be described as a

A. Cylinder

B. Sphere

C. Paraboloid

D. Half-Cone

E. None of the above.

12. (4 points) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx =$

A. $8\pi/3$

B. $16\pi/3$

C. π

D. $4\pi/3$

E. None of the above.

13. (4 points) Let D be the solid inside the cylinder $x^2 + (y-1)^2 = 1$, bounded below by the plane $z = 0$ and above by the paraboloid $z = x^2 + y^2$. Then the volume of D is given by the triple integral in cylindrical coordinates

$$\iiint_D dV = \int_0^{\pi} \int_0^{a} \int_0^{b} r\, dz\, dr\, d\theta$$

where

A. $a = 1$ and $b = r$

B. $a = \cos\theta$ and $b = r^2$

C. $a = 2\sin\theta$ and $b = r^2$

D. $a = 1$ and $b = r^2$

E. None of the above.

14. (4 points) Find the surface area of the portion of the plane $y + 2z = 2$ that lies inside the cylinder $x^2 + y^2 = 1$.

A. $\pi\sqrt{5}$

B. $2\pi\sqrt{5}$

C. $\sqrt{5}/2$

D. $\pi\sqrt{5}/2$

E. None of the above.

15. (4 points) Find the work done by the force $\mathbf{F} = \langle y, -xy \rangle$ along the straight-line segment from $(0, 0)$ to $(3, 1)$.

A. $5/2$

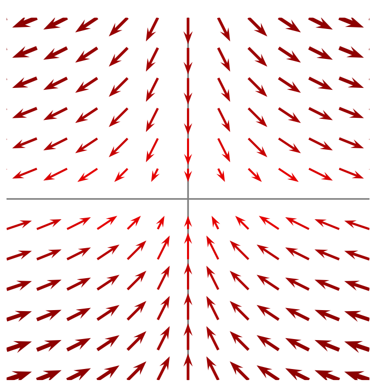
B. $-3/2$

C. $1/2$

D. $-1/2$

E. None of the above.

16. (4 points) Which of the following vector field plots could be $\mathbf{F} = xy\,\mathbf{i} - y\,\mathbf{j}$?

C. 

More Challenging Problem(s). Show all work to receive credit.

17. (4 points) Let f be a differentiable function of x and y with continuous second order partial derivatives on a domain D. Also, let $\mathbf{F} = \nabla f = M\,\mathbf{i} + N\,\mathbf{j}$ be a gradient field and C be any smooth curve connecting distinct points A and B be in D. Consider the following statements:

(a) $\mathbf{F}$ is a conservative vector field.

(b) $\displaystyle\int_C M\,dx + N\,dy = 0$

(c) $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

A. Only (a) is true.

B. Only (b) is true.

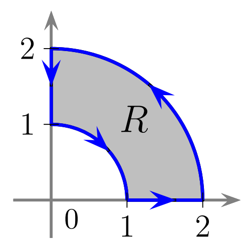
C. Only (c) is true.

D. Only (a) and (b) are true.

E. Only (a) and (c) are true.

F. Only (b) and (c) are true.

18. (10 points) Let C be the closed curve in the first quadrant consisting of the line segment from $(1,0)$ to $(2,0)$, the circle of radius 2 from $(2,0)$ to $(0,2)$, the line segment from $(0,2)$ to $(0,1)$ and the circle from $(0,1)$ to $(1,0)$, oriented counterclockwise (see the figure below). Compute the outward flux of the vector field $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$ across C.



Solution: By the normal form of Green's Theorem we have

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA$$
$$= \iint_R (2x + x)\,dA$$
$$= \int_0^{\pi/2}\int_1^2 3r^2\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/2}\cos\theta\,d\theta \int_1^2 3r^2\,dr$$
$$= 1 \times r^3\Big|_1^2 = 7$$

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1,0,0\rangle \quad \mathbf{j} = \langle 0,1,0\rangle \quad \mathbf{k} = \langle 0,0,1\rangle$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$$
- **Cross Product**:
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a,b,c\rangle$:
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a,b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\,dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}}f(x,y) = \nabla f\cdot\mathbf{u}$
- **Second Derivative Test** Suppose $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let
$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$
  - (a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
  - (b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
  - (c) If $D < 0$ then $f(a,b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x\cos x$

Multiple Integrals

- **Area**: $A(D) = \iint_D 1 \; dA$
- **Volume**: $V(E) = \iiint_E 1 \; dV$

Polar/Cylindrical

- Transformations

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \\ y/x &= \tan\theta \end{aligned}$$

- $\iint_D f(x,y) \; dA = \iint_D f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$
- $\iiint_E f(x,y,z) \; dV =$
  $\iiint_E f(r\cos\theta, r\sin\theta, z) \, r \, dz \, dr \, d\theta$

Spherical

- Transformations

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}$$

- $\iiint_E f(x,y,z) \; dV =$

$\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2\sin\phi) \, d\rho \, d\phi \, d\theta$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- $\mathbf{F}$ is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem (Tangential Form)**

$$\int_C P \; dx + Q \; dy = \iint_D (Q_x - P_y) \, dA$$

- **Green's Theorem (Normal Form)**

$$\int_C P \; dy - Q \; dx = \iint_D (P_x + Q_y) \, dA$$