Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 9.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (7 points) Find a parametrization of the line of intersection between planes $2x + z = 1$ and $y + z = 2$.

Solution: Let L denote the line of intersection of two planes. Find coordinates (x_0, y_0, z_0) of a point $P \in L$. Triple (x_0, y_0, z_0) satisfy both equations. Set $z_0 = 0$, then $2x_0 + z_0 = 1$ implies $x_0 = 1/2$. Equation $y_0 + z_0 = 2$ implies $y_0 = 2$. Point $P(1/2, 2, 0)$ is a reference point for line L

Normal vector to the first plane $\overrightarrow{n_1} = \langle 2, 0, 1\rangle$.

Normal vector to the second plane $\overrightarrow{n_2} = \langle 0, 1, 1\rangle$.

Direction vector of L can be chosen $\overrightarrow{v} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \langle -1, -2, 2\rangle$.

Parametric equations take the form $x(t) = 1/2 - t,\ y(t) = 2 - 2t,\ z(t) = 2t$.

2. (7 points) What is the length of the curve $\mathbf{r}(t) = \langle \sin(2t), t, \cos(2t)\rangle$ from $t = 0$ to $t = \pi$?

Solution: Length

$$\begin{aligned} L &= \int_0^\pi |\mathbf{r}'(t)|\ dt \\ &= \int_0^\pi \sqrt{(2\cos(2t))^2 + 1^2 + (-2\sin(2t)^2)}\, dt \\ &= \int_0^\pi \sqrt{5}\, dt = \sqrt{5}\pi \end{aligned}$$

3. (7 points) Evaluate the limit or show that it does not exist: $\lim\limits_{\substack{x\to 1\\ y\to -1}} \dfrac{\sqrt{x}-\sqrt{2+y}}{x-y-2}$

Solution: Simplify,

$$\begin{aligned}
&\lim_{\substack{x\to 1\\ y\to -1}} \frac{\sqrt{x}-\sqrt{2+y}}{x-(y+2)}\\
&\lim_{\substack{x\to 1\\ y\to -1}} \frac{\sqrt{x}-\sqrt{2+y}}{x-(y+2)} \cdot \frac{\sqrt{x}+\sqrt{2+y}}{\sqrt{x}+\sqrt{2+y}}\\
&= \lim_{\substack{x\to 1\\ y\to -1}} \frac{1}{\sqrt{x}+\sqrt{2+y}}\\
&= \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

4. (7 points) Evaluate the limit or show that it does not exist: $\lim\limits_{\substack{x\to 0^+\\ y\to 0^+}} \dfrac{4x^2+9y^2}{2xy}$

Solution: We will show that the limit does not exist. Approach the origin along the line $y = mx$.

$$\begin{aligned}
\lim_{x\to 0} \frac{4x^2+9(mx)^2}{2x(mx)} &= \lim_{x\to 0} \frac{4+9m^2}{2m}\\
&= \frac{4+9m^2}{2m}
\end{aligned}$$

Note the the previous expression is $13/2$ if $m = 1$ and 10 for $m = 2$.

Since $13/2 \neq 10$ the limit does not exist.

5. Consider the function $f(x, y) = \sqrt{x^2 + 6xy + 9y^2}$ in the first quadrant.

(a) (5 points) Calculate the partial derivatives, f_x and f_y.

Solution: Note for $x > 0,\ y > 0$ we have

$$f(x, y) = \sqrt{(x + 3y)^2} = x + 3y$$

Therefore

$$\begin{aligned} f_x(x, y) &= 1 \\ f_y(x, y) &= 3 \end{aligned}$$

(b) (5 points) Use your answer in part (a) to find the linearization of f at $(1, 2)$.

Solution: We have $f(1, 2) = 7$ and given the results from (a) we get

$$\begin{aligned} L(x, y) &= 7 + 1(x - 1) + 3(y - 2) \\ &= x + 3y \end{aligned}$$

(c) (4 points) Use your answer in part (b) to approximate $f(0.7, 2.1)$.

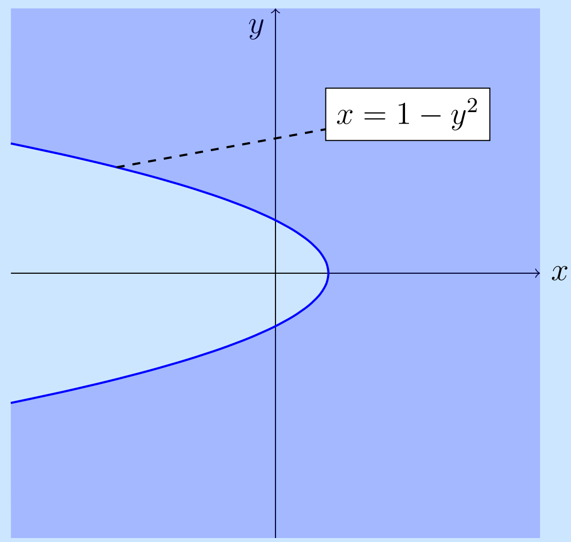
Solution:

$$\begin{aligned} f(0.7, 2.1) &\approx L(0.7, 2.1) \\ &\approx 0.7 + 3(2.1) \\ &\approx 0.7 + 6.3 \\ &\approx 7 \end{aligned}$$

6. Consider the function $f(x,y) = 2 - \sqrt{y^2 + x - 1}$

(a) (5 points) Sketch the domain of f.

Solution: Domain is determined by inequality
$D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1 - y^2\}$



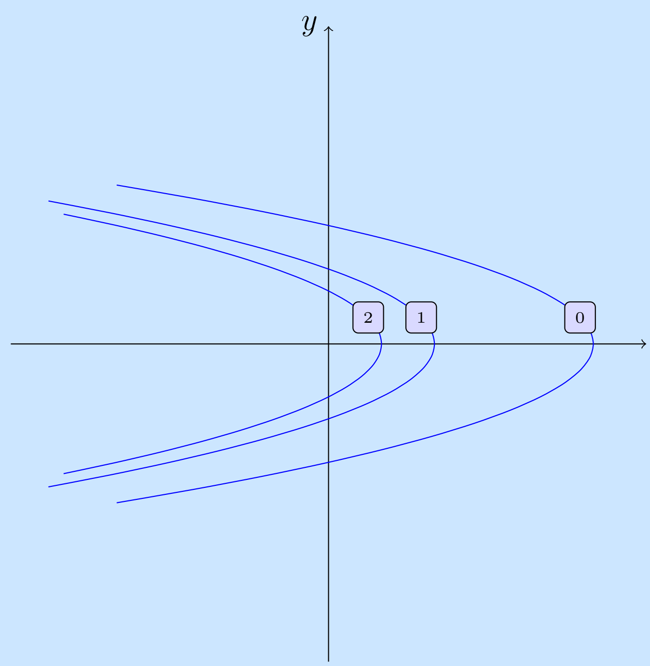
(b) (6 points) Sketch the level curves of f for levels $k = 0$, $k = 1$, and $k = 2$.

Solution:

for $k = 0$

$$\begin{aligned} 0 &= 2 - \sqrt{y^2 + x - 1} \\ \sqrt{y^2 + x - 1} &= 2 \\ y^2 + x - 1 &= 4 \\ x &= 5 - y^2 \end{aligned}$$

similarly for $k = 1$ we get $x = 2 - y^2$
and $k = 2$ we get $x = 1 - y^2$.



(c) (3 points) What is the range of f?

Solution: Range $R = (-\infty, 2]$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) The intersection of the quadric surfaces defined by $x^2 + y^2 + z^2 = 4$ and $x^2 + z^2 = 4$ is:
   - A. one point
   - B. two points
   - C. a circle
   - D. a straight line
   - E. a parabola

8. (4 points) Find the equation for the sphere in standard form centered at $(1, 2, 3)$ with radius 3.
   - A. $(x + 1)^2 + (y + 2)^2 + (z + 3)^2 = 9$
   - B. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 8$
   - C. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$
   - D. $(x - 1) + (y - 2) + (z - 3) = 3$
   - E. $x + 2y + 3z = 3$

9. (4 points) What is the positive value of parameter t corresponding to a point where the line defined by $(4t, 3t, 1)$ intersects the cone $z^2 = x^2 + y^2$?
   - A. 5/2
   - B. 3/10
   - C. 8/5
   - D. 1/5
   - E. 15

10. (4 points) Find the area of the triangle formed by the points $O(0,0,0)$, $P(1,0,2)$ and $Q(2,-1,-1)$

A. 4
B. $\sqrt{15/2}$
C. $\sqrt{11}$
D. $\sqrt{10}$
E. 3

11. (4 points) Which of the following is a parametrization for the full curve of intersection between $x^2+y^2-z^2=0$ and $z=x+1$

A. $\mathbf{r}(t) = \left\langle t, \sqrt{2t+1}, t+1 \right\rangle$ with $t \in [-1/2, \infty)$
B. $\mathbf{r}(t) = \left\langle \frac{t^2-1}{2}, t, \frac{t^2+1}{2} \right\rangle$ with $t \in [0, \infty)$
C. $\mathbf{r}(t) = \left\langle \frac{t^2-1}{2}, t, \frac{t^2+1}{2} \right\rangle$ with $t \in (-\infty, \infty)$
D. $\mathbf{r}(t) = \langle \cos(t), \sin(t), -2 \rangle$ with $t \in [0, 2\pi]$
E. $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ with $t \in [0, 2\pi]$

12. (4 points) Which of the following is the unit tangent vector to the curve: $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{2}{3}t^3 \right\rangle$ at $\mathbf{r}(1) = \langle 2, 1, 2/3 \rangle$

A. $\mathbf{r}(t) = \langle 2, 1, 2/3 \rangle$
B. $\mathbf{r}(t) = \langle 2, 2, 2 \rangle$
C. $\mathbf{r}(t) = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$
D. $\mathbf{r}(t) = \langle 2, 2t, 2t^2 \rangle$
E. $\mathbf{r}(t) = \langle 2, 0, -\cos(t) \rangle$

13. (4 points) Consider the function $f(x, y) = \ln(y)x$. Calculate $f_{xx} + f_{xy}$.

A. $f_{xx} + f_{xy} = \frac{1}{y} + 1$

B. $f_{xx} + f_{xy} = \frac{1}{y}$

C. $f_{xx} + f_{xy} = 0$

D. $f_{xx} + f_{xy} = 1$

E. $f_{xx} + f_{xy} = \frac{1}{x^2y^2} - \frac{2}{x^3y}$

14. (4 points) Suppose $f(x, y) = x^2y$ where $x(t) = 3\cos(t) + 2\sin(t)$ and $y(t) = 5\cos(t) - 4\sin(t)$. Which of the following is equal to $\frac{df}{dt}$ at $t = 0$?

A. 6

B. 24

C. -14

D. -15

E. -3

15. (4 points) Find the vector **a** with $|\mathbf{a}| = \mathbf{6}$ orthogonal to the plane given by $x + 2y + 2z + 5 = 0$.

A. $\langle 0, 3, -1\rangle$

B. $\left\langle 0, \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right\rangle$

C. $\langle 2, 0, t^2\rangle$

D. $\langle 1, 2, 2\rangle$

E. $\langle 2, 4, 4\rangle$

More Challenging Problem(s). Show all work to receive credit.

16. (7 points) Find the distance between the two skew lines given by vector equations

$$\langle 1+2t, 0, t\rangle \qquad \text{and} \qquad \langle -1+t, 2+t, 1+t\rangle$$

Solution: Direction vector for the first line $\overrightarrow{v_1} = \langle 2, 0, 1\rangle$.

Direction vector for the second line $\overrightarrow{v_2} = \langle 1, 1, 1\rangle$.

Vector orthogonal to both lines $\overrightarrow{n} = \overrightarrow{v_1} \times \overrightarrow{v_2} = \langle -1, -1, 2\rangle$.

Point $P(1, 0, 0)$ lies on the first line, $R(-1, 2, 1)$ lies on the second line.
The distance between lines can be given by the formula

$$\begin{aligned} D &= |\text{comp}_{\vec{n}} \overrightarrow{PR}| \\ &= \frac{|\vec{n} \cdot \overrightarrow{PR}|}{|\vec{n}|} \\ &= \frac{|\,\langle -1, -1, 2\rangle \cdot \langle -2, 2, 1\rangle\,|}{|\,\langle -1, -1, 2\rangle\,|} = \frac{2-2+2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \end{aligned}$$

17. (7 points) An object moves along the surface $x^2 + z^2 = 9$ in $\mathbb{R}^3$ via the parametric equations:

$$x(t) = 3\cos(\pi t) \qquad \text{and} \qquad y(t) = 4\pi t$$

What is the speed and of the object at time $t = 2$?

Solution: There are two option for parametrized curve:

$x(t) = 3\cos(\pi t),\ y(t) = 4\pi t,\ z(t) = 3\sin(\pi t)$ or

$x(t) = 3\cos(\pi t),\ y(t) = 4\pi t\ z(t) = -3\sin(\pi t)$.

In both cases, the speed equals

$$\begin{aligned} s(t) &= |\mathbf{r}'(t)| \\ &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \\ &= \sqrt{(-3\pi\sin(\pi t))^2 + (4\pi)^2 + (\pm 3\pi\cos(\pi t))^2} \\ &= \sqrt{9\pi^2 + 16\pi^2} \\ &= 5\pi \end{aligned}$$

Notice that this is true for all time, so it is true at $t = 2$.