Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page for scratch work. **Work on the back of pages will not be graded.**
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (14 points) Consider the function $f(x, y) = 2x^2 - 4xy + y^4 + 2$

   (a) Find the critical points of f and classify them as local minima, local maxima, or saddle points.

   Solution:

   $$f_x = 4x - 4y \qquad\qquad f_y = -4x + 4y^3$$

   which yield critical points at $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

   $$f_{xx} = 4 \qquad\qquad f_{xy} = -4 \qquad\qquad f_{yy} = 12y^2$$

   - at $(0, 0)$

     $D = (4)(0) - (-4)^2 = -16 < 0$ so $(0, 0)$ is a saddle point.
   - at $(1, 1)$

     $D = (4)(12) - (-4)^2 = 32 > 0$ and $f_{xx} = 4$so $(1, 1)$ is a local min.
   - at $(-1, -1)$

     $D = (4)(12) - (-4)^2 = 32 > 0$ and $f_{xx} = 4$so $(-1, -1)$ is a local min.

   (b) Let C be the curve $y = x$ for $x \in [0, 3]$. What is the absolute minimum and maximum of f on C?

   Solution: $f(x, x) = 2x^2 - 4x^2 + x^4 + 2 = x^4 - 2x^2 + 2$ and so $f'(x) = 4x^3 - 4x$ which has critical points at $x = 0$ and $x = 1$ on the interval $x \in [0, 3]$. We have

   $$\begin{aligned} f(0) &= 2 \\ f(1) &= 1 - 2 + 2 = 1 \\ f(3) &= 81 - 18 + 2 = 65 \end{aligned}$$

   So 1 is the absolute min of f on C and 65 is the absolute max of f on C.

2. (14 points) Evaluate

$$\int_C \big((x+y)\mathbf{i} + (y-z)\mathbf{j} + z^2\mathbf{k}\big) \cdot d\mathbf{r}$$

where C is given by vector function $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$.

Solution: $\mathbf{r}'(t) = \langle 2t, 3t^2, 2t\rangle$ and so

$$\begin{aligned}
\int_C \big((x+y)\mathbf{i} + (y-z)\mathbf{j} + z^2\mathbf{k}\big) \cdot d\mathbf{r} &= \int_0^1 \big\langle x+y, y-z, z^2\big\rangle \cdot \big\langle 2t, 3t^2, 2t\big\rangle \; dt \\
&= \int_0^1 \big\langle t^2+t^3, t^3-t^2, (t^2)^2\big\rangle \cdot \big\langle 2t, 3t^2, 2t\big\rangle \; dt \\
&= \int_0^1 (2t^3+2t^4) + (3t^5-3t^4) + (2t^5) \; dt \\
&= \int_0^1 5t^5 - t^4 + 2t^3 \; dt \\
&= \left[\frac{5}{6}t^6 - \frac{1}{5}t^5 + \frac{1}{2}t^4\right]_0^1 \\
&= \frac{5}{6} - \frac{1}{5} + \frac{1}{2}
\end{aligned}$$

3. (14 points) Find the surface area of the surface given by the graph $z = 1 + 3x + \ln\left(\sqrt{y}\right) - \frac{5}{2}y^2$ over the domain $1 \le x \le 4$, $1 \le y \le 2$.

Solution:

$$\begin{aligned} f_x &= 3 \\ f_y &= \frac{1}{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} - 5y = \frac{1}{2y} - 5y = \frac{1-10y^2}{2y} \end{aligned}$$

and so the surface area can be given by

$$\begin{aligned} \int_1^4 \int_1^2 \sqrt{(3)^2 + \left(\tfrac{1-10y^2}{2y}\right)^2 + 1}\; dy\; dx &= \int_1^4 \int_1^2 \sqrt{10 + \left(\tfrac{1-20y^2+100y^4}{4y^2}\right)}\; dy\; dx \\ &= \int_1^4 \int_1^2 \sqrt{\tfrac{40y^2}{4y^2} + \left(\tfrac{1-20y^2+100y^4}{4y^2}\right)}\; dy\; dx \\ &= \int_1^4 \int_1^2 \sqrt{\left(\tfrac{1+20y^2+100y^4}{4y^2}\right)}\; dy\; dx \\ &= \int_1^4 \int_1^2 \left(\frac{1+10y^2}{2y}\right)\; dy\; dx \\ &= \int_1^4 1\; dx \cdot \int_1^2 \left(\frac{1}{2y} + 5y\right)\; dy \\ &= [x]_1^4 \cdot \left[\tfrac{1}{2}\ln y + \tfrac{5}{2}y^2\right]_1^2 \\ &= 3 \cdot \left[\tfrac{1}{2}\ln 2 + \tfrac{5}{2}(3)\right] \\ &= \frac{3}{2}\ln 2 + \frac{45}{2} = \frac{3}{2}(\ln 2 + 15) \end{aligned}$$

4. (14 points) Consider the vector field $\mathbf{F}(x,y,z) = \left(2xy^2z^3\right)\mathbf{i} + \left(2x^2yz^3\right)\mathbf{j} + \left(3x^2y^2z^2\right)\mathbf{k}$

(a) Compute the curl($\mathbf{F}$) (show your calculations)

Solution:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z^3 & 2x^2yz^3 & 3x^2y^2z^2 \end{vmatrix} = (6x^2yz^2 - 6x^2yz^2)\mathbf{i} - (6xy^2z^2 - 6xy^2z^2)\mathbf{j} + (4xyz^3 - 4xyz^3)\mathbf{k} = \langle 0,0,0\rangle$$

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Solution:

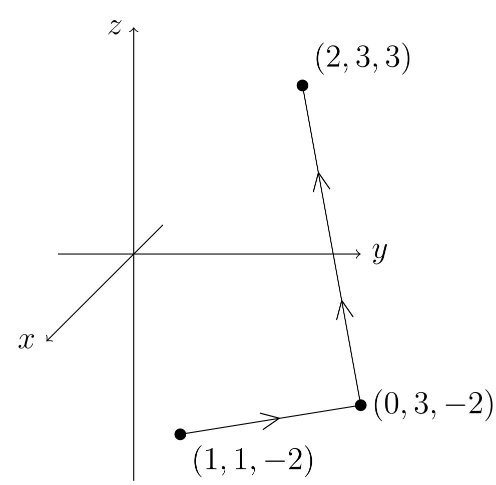
$$f = \int 2xy^2z^3\;dx$$
$$f = x^2y^2z^3 + g(y,z)$$
$$f_y = 2x^2yz^3 + g_y(y,z) = 2x^2yz^3$$

so $g_y = 0$ giving $g(y,z) = h(z)$

$$f_z = 3x^2y^2z^2 + h'(z) = 3x^2y^2z^2$$

and so $h'(z) = 0$ giving us $h(z) = K$ so a possible final solution is $f = x^2y^2z^3$.

(c) Evaluate a line integral $\int_C \mathbf{F}\cdot\mathbf{T}\;ds$ where C is the curve shown below.



Solution:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\;ds &= f(2,3,3) - f(1,1,-2)\\ &= (4)(9)(27) - (1)(1)(-8)\\ &= 972 + 8 = 980\end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

5. (4 points) E is the cube given by $0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a$ with mass density $\sigma(x, y, z) = x^2 + y^2 + z^2$. The mass is given by the formula:

   A. $\int_0^a \int_0^a \int_0^{\sqrt{1-x^2-y^2}} 1\ dz\ dy\ dx$

   B. $\int_0^a \int_0^a \int_0^{\sqrt{1-x^2-y^2}} x^2 + y^2 + z^2\ dz\ dy\ dx$

   C. $\int_0^a \int_0^a \int_0^a 1\ dz\ dy\ dx$

   D. $\int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2\ dz\ dy\ dx$

   E. None of the above.

6. (4 points) Which of the following is the equations of the tangent plane to the surface $xyz^2 = 6$ at the point $(3, 2, 1)$.

   A. $x + y = 3$

   B. $2x + 3y + 12y = 0$

   C. $2x - 3y + 6y = 24$

   D. $2x + 3y + 12z = 24$

   E. $y + 3z = 24$

7. (4 points) Evaluate $\iiint_B \frac{1}{\sqrt{x^2 + y^2}}\, dV$, where B lies between the spheres

$$x^2 + y^2 + z^2 = 4 \text{ and } x^2 + y^2 + z^2 = 9$$

   A. 3π

   B. $-\pi^2$

   C. $5\pi^2$

   D. π

   E. π^2

8. (4 points) Reversing the order of integration $\int_2^4 \int_1^{9-2y} 3x + y \; dx \; dy =$

   A. $\int_2^6 \int_5^{y-1} 3x + y \; dy \; dx$

   B. $\int_1^5 \int_2^{(9-x)/2} 3x + y \; dy \; dx$

   C. $\int_1^5 \int_5^{2-x} 3x + y \; dy \; dx$

   D. $\int_2^4 \int_1^5 3x + y \; dy \; dx$

   E. $\int_1^{9-2y} \int_2^4 3x + y \; dy \; dx$

9. (4 points) If $\mathbf{F}(x, y, z) = z\mathbf{i} + (2x + z)\mathbf{j} + 3\mathbf{k}$, then $\text{div}(\mathbf{F}) =$

   A. 0

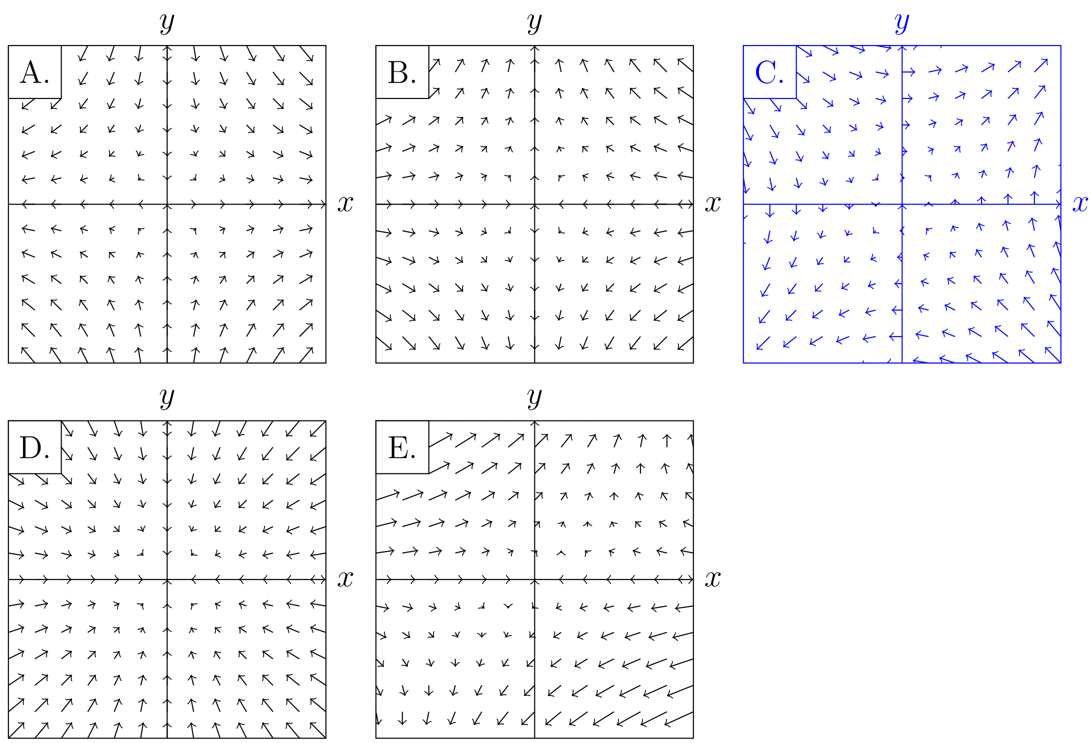
   B. $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

   C. $1 + x + y$

   D. xz

   E. $x^2 + y + 2z$

10. (4 points) Which of the following could be the gradient vector field ∇f of the function $f(x, y) = xy$.



11. (4 points) Evaluate $\int\limits_C \langle -2y, y\rangle \cdot d\mathbf{r}$, where C is the line segment from $(0,0)$ to $(1,2)$.

A. 3
B. 4
C. 5
D. 0
E. 9

12. (4 points) The area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, with $0 \le u \le 1$ and $0 \le v \le 2$

A. 1
B. 2
C. 3
D. 4
E. 5

13. (4 points) Consider the closed curve C which is the union of three paths; the straight line from $(0,0)$ to $(1,0)$, the arc of the unit circle from $(1,0)$ to $(0,1)$, and finally, straight line from $(0,1)$ to $(0,0)$. Evaluate $\int_C (x+y)dx - (x-y)dy$

A. -2
B. 3
C. $-\pi/2$
D. $-\pi/4$
E. $-\pi^2$

More Challenging Problem(s). Show all work to receive credit.

14. (8 points) Consider the vector field $\mathbf{F} = \dfrac{x-y}{x^2+y^2}\mathbf{i} + \dfrac{Ax+y}{x^2+y^2}\mathbf{j}$

(a) Compute the value of A so that the vector field is conservative on its domain.

Solution:

$$Q_x = \frac{A(x^2+y^2)-(Ax+y)(2x)}{(x^2+y^2)^2} \qquad P_y = \frac{(-1)(x^2+y^2)-(x-y)(2y)}{(x^2+y^2)^2}$$

So since we want $Q_x = P_y$ we get

$$\begin{aligned} Ax^2 + Ay^2 - 2Ax^2 - 2xy &= -x^2 - y^2 - 2xy + 2y^2 \\ Ay^2 - Ax^2 &= -x^2 + y^2 \end{aligned}$$

Which yields $A = 1$.

(b) For this value of A, explain the equality $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_S \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle centered at the origin and S is the boundary of the square $[-1,1]\times[-1,1]$.

Solution: Consider the region D inside the square and outside the circle on which $\mathbf{F}$ is continuous. Using Green's Theorem we have

$$\oint_S \mathbf{F}\cdot d\mathbf{r} - \oint_C \mathbf{F}\cdot d\mathbf{r} \overset{\text{GT}}{=} \iint_D Q_x - P_y \; dA = \iint 0 \; dA = 0$$

and so $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_S \mathbf{F}\cdot d\mathbf{r}$

15. (6 points) Let the vector field $\mathbf{F}(x,y)$ in the plane satisfies relation $\mathbf{F}(x,y) = \mathbf{F}(-x,-y)$. Show that the line integral $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ where C is the boundary of the square $[-1,1]\times[-1,1]$.

Solution: Consider the following definitions: $\mathbf{F} = \langle P, Q\rangle$.

$$\mathbf{L}(t) = \langle -1, t\rangle\,,\ t\in[1,-1], \quad \mathbf{R}(t) = \langle 1, t\rangle\,,\ t\in[-1,1], \quad \mathbf{B}(t) = \langle t, -1\rangle\,,\ t\in[-1,1], \quad \mathbf{T}(t) = \langle t, 1\rangle\,,\ t\in[1,-1]$$

which parametrize the sides of the square. Then

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_R \langle P,Q\rangle\cdot d\mathbf{R} + \int_T \langle P,Q\rangle\cdot d\mathbf{T} + \int_B \langle P,Q\rangle\cdot d\mathbf{B} + \int_L \langle P,Q\rangle\cdot d\mathbf{L} \\ &= \int_{-1}^{1} \langle P(1,t), Q(1,t)\rangle\cdot\langle 0,1\rangle\, dt + \int_{1}^{-1} \langle P(t,1), Q(t,1)\rangle\cdot\langle 1,0\rangle\, dt \\ &\quad + \int_{-1}^{1} \langle P(t,-1), Q(t,-1)\rangle\cdot\langle 1,0\rangle\, dt + \int_{1}^{-1} \langle P(-1,t), Q(-1,t)\rangle\cdot\langle 0,1\rangle\, dt \\ &= \int_{-1}^{1} Q(1,t)\; dt + \int_{1}^{-1} P(t,1)\; dt + \int_{-1}^{1} P(t,-1)\; dt + \int_{1}^{-1} Q(-1,t)\; dt \\ &= \int_{-1}^{1} Q(1,t)\; dt - \int_{-1}^{1} P(t,1)\; dt + \int_{-1}^{1} P(t,-1)\; dt - \int_{-1}^{1} Q(-1,t)\; dt = 0 \end{aligned}$$

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are $\boxed{\text{BOXED}}$.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET PAGE 1

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= |\mathbf{u}||\mathbf{v}| \cos\theta \end{aligned}$$
- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\quad \text{proj}_{\mathbf{u}} \mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\quad \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)| \, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u}$
- **Second Derivative Test** Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
  (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  (c) If $D < 0$ then $f(a, b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

FORMULA SHEET PAGE 2

Multiple Integrals

- **Area**: $A(D) = \iint_D 1 \; dA$
- **Volume**: $V(E) = \iiint_E 1 \; dV$

Polar/Cylindrical

- Transformations

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \\ y/x &= \tan\theta \end{aligned}$$

- $\iint_D f(x,y) \; dA = \iint_D f(r\cos\theta, r\sin\theta) \; r \; dr \; d\theta$
- $\iiint_E f(x,y,z) \; dV = \iiint_E f(r\cos\theta, r\sin\theta, z) \; r \; dz \; dr \; d\theta$

Spherical

- Transformations

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}$$

- $\iint_E f(x,y,z) \; dV =$
  $\iint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2\sin\phi) \; d\rho \, d\phi \, d\theta$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- $\mathbf{F}$ is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem**

$$\int_C P \; dx + Q \; dy = \iint_D (Q_x - P_y) \, dA$$