1. (7 points) Let $f(x,y) = x^2+3xy+y^2+y$. Find and classify each critical point of f as a local minimum, a local maximum, or a saddle point.

Solution:

$$f_x = 2x+3y \qquad f_y = 3x+2y+1$$
$$f_{xx} = 2 \qquad f_{xy} = 3 \qquad f_{yy} = 2$$

Both $f_x = 0$ and $f_y = 0$ only at $(-3/5, 2/5)$. Using the second derivative test we have

$$D = 2(2) - 3^2 = 4 - 9 = -5 < 0$$

So therefore $(-3/5, 2/5)$ is a saddle point.

2. (7 points) Express the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2+y^2+z^2 = 4$ as an integral using cylindrical coordinates and evaluate it.

Solution: The sphere can be expressed as $z = \pm\sqrt{4-x^2-y^2} = \pm\sqrt{4-r^2}$. So the volume can be expressed in cylindrical coordinates as:

$$\begin{aligned} V = \iiint 1\ dV &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta \\ &= 2\pi\int_0^1 2r\sqrt{4-r^2}\ dr \\ &= 4\pi\left[\tfrac{-1}{3}(4-r^2)^{3/2}\right]_0^1 \\ &= \frac{4\pi}{3}\left[8-3^{3/2}\right] \end{aligned}$$

⋆

3. Let $\mathbf{F}(x, y) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and answer the following questions

(a) (7 points) Find a function f so that $\nabla f = \mathbf{F}$.

Solution: One possible answer is $f = xyz + z^2$. Can check that

$$f_x = yz \qquad f_y = xz \qquad f_z = xy + 2z$$

(b) (4 points) Let C be a curve given by vector equation $C : \mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{2t}{\pi}\mathbf{k}$, $t \in [0, \pi/2]$. Evaluate the integral

$$\int_C yz\ dx + xz\ dy + (xy + 2z)\ dz.$$

Solution: $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ and $\mathbf{r}(\pi/2) = \langle 0, 1, 1\rangle$ so by the fundamental theorem of line integrals

$$\begin{aligned}\int_C yz\ dx + xz\ dy + (xy + 2z)\ dz &= f(0, 1, 1) - f(1, 0, 0) \\ &= 1 - 0 = 1\end{aligned}$$

(c) (3 points) Let line segment C_1 in $\mathbb{R}^3$ go from $P(1, 0, 0)$ to $R(1, 1, 1)$, and line segment C_2 go from R to $Q(0, 1, 1)$. Evaluate the integral.

$$\int_{C_1 \cup C_2} yz\ dx + xz\ dy + (xy + 2z)\ dz.$$

Solution: Because the paths start and end at the same points as (b) we know conservative vector field have path independent line integrals so the answer must also be 1.

4. (7 points) The solid D lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 4$.

Use spherical coordinates to evaluate the integral $\iiint_D z \, dV$ of the height function z over the solid D.

Solution: The sphere can be expressed as $\rho^2 = 4 \implies \rho = 2$. The cone can be expressed as

$$\begin{aligned} z &= \sqrt{r^2} \\ z &= r \\ \rho \cos\phi &= \rho \sin\phi \\ \cos\phi &= \sin\phi \\ \phi &= \pi/4 \end{aligned}$$

So

$$\begin{aligned} \iiint z \, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta \\ &= 2\pi \left[\frac{\sin^2\phi}{2} \right]_0^{\pi/4} \left[\frac{\rho^4}{4} \right]_0^2 \\ &= 2\pi \left[\frac{1}{4} \right] [4] = 2\pi \end{aligned}$$

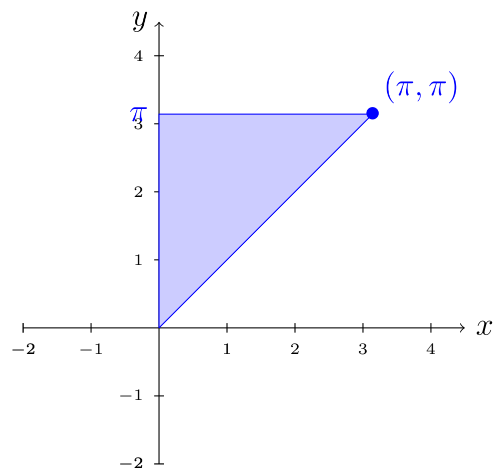
5. (7 points) Use Green's theorem to find the work done by the force $\mathbf{F} = (x - 3y)\mathbf{i} + (y - x)\mathbf{j}$ on a particle moving counter-clockwise around the circle $(x - 2)^2 + y^2 = 4$.

Solution:

$$\begin{aligned} Work = \int_C \mathbf{F} \cdot \mathbf{T} \, ds &= \iint_D (y - x)_x - (x - 3y)_y \, dA \\ &= \iint_D -1 - (-3) \, dA \\ &= 2 \iint_D 1 \, dA \qquad \text{(area of the circle)} \\ &= 2(\pi 2^2) = 8\pi \end{aligned}$$

6. Consider the integral $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$

(a) (3 points) Sketch the region of integration.
Label all relevant intersection points.



(b) (5 points) Evaluate the integral above by reversing the order of integration.

Solution:

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx = \int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy$$
$$= \int_0^\pi \sin y\, dy$$
$$= -\cos(y)\Big|_0^\pi = 2$$

7. (6 points) Find the surface area of the part of the half-cone $z = \sqrt{x^2+y^2}$ bounded from above by the plane $z = 1$.

Solution:

$$z_x = \frac{x}{\sqrt{x^2+y^2}} \qquad z_y = \frac{y}{\sqrt{x^2+y^2}}$$

So surface area is given by

$$\iint_D \sqrt{(z_x)^2 + (z_y)^2 + 1}\, dA = \iint_D \sqrt{\left(\frac{x^2}{x^2+y^2}\right) + \left(\frac{y^2}{x^2+y^2}\right) + \frac{x^2+y^2}{x^2+y^2}}\, dA$$
$$= \iint_D \sqrt{2}\, dA$$
$$= \sqrt{2} \iint_D dA$$

The cone is over the circle of radius 1 centered at the origin in the xy-plane. So using the area of a circle formula we have the final result

$$= \sqrt{2}(\pi 1^2) = \sqrt{2}\pi$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) The surface $\rho = 2\cos\phi$ can be described as a

   A. Plane

   B. Half-Cone

   C. Sphere

   D. Paraboloid

   E. None of the above

9. (4 points) Evaluate $\iiint_E y \, dV$, where $E = \{(x, y, z) \mid 0 \leq x \leq 3,\ 0 \leq y \leq x,\ x - y \leq z \leq x + y\}$.

   A. 9

   B. $\frac{27}{2}$

   C. $-\frac{9}{2}$

   D. $\frac{5}{3}$

   E. None of the above

10. (4 points) Find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 3t\mathbf{k},\ 0 \leq t \leq 2$, if the density is $\delta = 4z$.

   A. 50

   B. 115

   C. 125

   D. 6π

   E. None of the above

11. (4 points) Consider the function $f(x, y) = x^2 + xy + y^2$ at the point $(-1, 1)$.
In what direction does f decrease most rapidly?

A. $\frac{1}{\sqrt{2}}\langle 1, 1\rangle$

B. $\frac{1}{\sqrt{2}}\langle 1, -1\rangle$

C. $\frac{1}{\sqrt{2}}\langle -1, 1\rangle$

D. $\frac{1}{\sqrt{2}}\langle -1, -1\rangle$

E. None of the above

12. (4 points) Find the work done by the force $\mathbf{F} = \langle xy, y, -yz\rangle$ over the curve $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t\mathbf{k}$, $0 \le t \le 1$ in the direction of increasing t.

A. 1/4

B. 1/3

C. 1/2

D. 1

E. None of the above

13. (4 points) The vector field below could have been generated by which of the following?

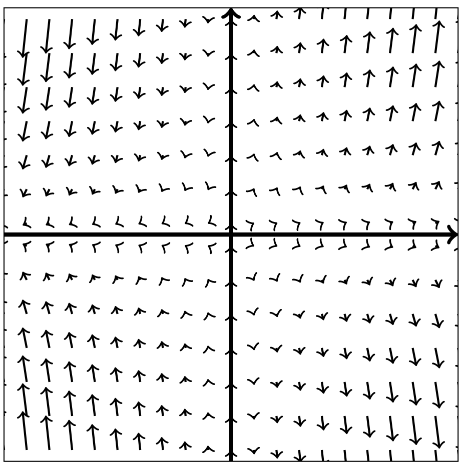
A. $x\mathbf{i} + xy\mathbf{j}$

B. $y\mathbf{i} - x^2\mathbf{j}$

C. $-y\mathbf{i} - xy\mathbf{j}$

D. $x\mathbf{i} - y^2\mathbf{j}$

E. $x^2\mathbf{i} - (y - x)\mathbf{j}$



14. (4 points) Let $\mathbf{F} = \langle 2x + y, x + z, y \rangle$. Which of the following is true?

A. curl $\mathbf{F} = 2x + y$ and div $\mathbf{F} = z - y + x$.

B. curl $\mathbf{F} = \langle 2, 0, 0 \rangle$ and div $\mathbf{F} = 0$.

C. curl $\mathbf{F} = \langle 0, 0, 0 \rangle$ and div $\mathbf{F} = 2$.

D. curl $\mathbf{F} = \langle z, -y, x + y \rangle$ and div $\mathbf{F} = 2x + y$.

E. None of the above

15. (4 points) Find the absolute maximum and minimum values of $f(x, y) = x^2 - 2x + y^2$ on the set $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

A. $\min = -1,\ \max = 1$

B. $\min = -4,\ \max = 4$

C. $\min = -1,\ \max = 8$

D. $\min = 0,\ \max = 8$

E. None of the above

16. (4 points) Estimate the change of the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ if the point $P(x, y, z)$ moves from $P_0(1, 1, 2)$ a distance of $ds = \frac{1}{5}$ units in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

A. 1

B. $-\frac{1}{\sqrt{5}}$

C. $\frac{1}{\sqrt{5}}$

D. $\frac{1}{21}$

E. None of the above

More Challenging Problem(s). Show all work to receive credit.

17. (6 points) Find the average value of the function $f(x) = \int_x^1 \cos(t^2)\,dt$ on the interval $[0, 1]$.

Solution: For 1-variable functions over intervals (calc 1) we have the average value of a function is

$$\begin{aligned} f_{ave} &= \frac{1}{1-0}\int_0^1 f\ dx \\ &= \int_0^1 \int_x^1 \cos(t^2)\ dt\ dx \end{aligned}$$

There is no closed form integral of $\cos(t^2)$ so we need to be clever. Through drawing a picture of the region and switching the bounds of integration

$$\begin{aligned} &= \int_0^1 \int_0^t \cos(t^2)\ dx\ dt \\ &= \int_0^1 t\cos(t^2)dt \qquad \text{(now use } u\text{-sub)} \\ &= \left[\frac{\sin(t^2)}{2}\right]_0^1 = \frac{\sin 1}{2} \end{aligned}$$

18. **TRUE or FALSE**? Circle the right choice. No work needed

(a) (2 points) If **F** and **G** are vector fields, then $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}(\mathbf{F}) + \text{curl}(\mathbf{G})$.

A. TRUE

B. FALSE

(b) (2 points) If **F** and **G** are vector fields, then $\text{curl}(\mathbf{F} \cdot \mathbf{G}) = \text{curl}(\mathbf{F}) \cdot \text{curl}(\mathbf{G})$.

A. TRUE

B. FALSE

(c) (2 points) $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\, dx\, dy = 0$

A. TRUE

B. FALSE

(d) (2 points) The integral $\int_0^{2\pi} \int_0^2 \int_r^2 dz\, dr\, d\theta = 0$ represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

A. TRUE

B. FALSE