Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page for scratch work. **Work on the back of pages will not be graded.**
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (7 points) Let $f(x, y) = x^2 + 3xy + y^2 + y$. Find and classify each critical point of f as a local minimum, a local maximum, or a saddle point.

2. (7 points) Express the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$ as an integral using cylindrical coordinates and evaluate it.

3. Let $\mathbf{F}(x, y) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and answer the following questions

(a) (7 points) Find a function f so that $\nabla f = \mathbf{F}$.

(b) (4 points) Let C be a curve given by vector equation $C : \mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{2t}{\pi}\mathbf{k}$, $t \in [0, \pi/2]$. Evaluate the integral

$$\int_C yz\ dx + xz\ dy + (xy + 2z)\ dz.$$

(c) (3 points) Let line segment C_1 in $\mathbb{R}^3$ go from $P(1, 0, 0)$ to $R(1, 1, 1)$, and line segment C_2 go from R to $Q(0, 1, 1)$. Evaluate the integral.

$$\int_{C_1 \cup C_2} yz\ dx + xz\ dy + (xy + 2z)\ dz.$$

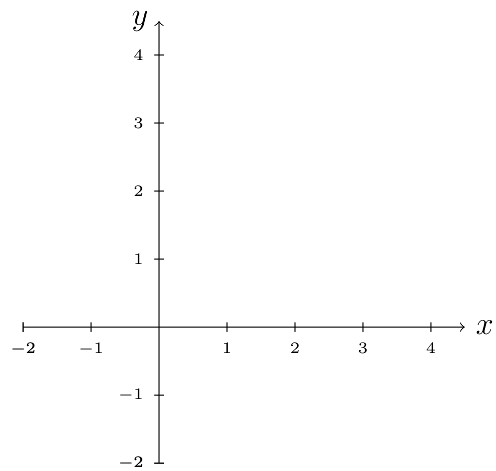
4. (7 points) The solid D lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 4$.

   Use spherical coordinates to evaluate the integral $\iiint_D z \, dV$ of the height function z over the solid D.

5. (7 points) Use Green's theorem to find the work done by the force $\mathbf{F} = (x - 3y)\mathbf{i} + (y - x)\mathbf{j}$ on a particle moving counter-clockwise around the circle $(x - 2)^2 + y^2 = 4$.

6. Consider the integral $\displaystyle\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx$

   (a) (3 points) Sketch the region of integration.
   Label all relevant intersection points.



   (b) (5 points) Evaluate the integral above by reversing the order of integration.

7. (6 points) Find the surface area of the part of the half-cone $z=\sqrt{x^2+y^2}$ bounded from above by the plane $z=1$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) The surface $\rho = 2\cos\phi$ can be described as a
   - A. Plane
   - B. Half-Cone
   - C. Sphere
   - D. Paraboloid
   - E. None of the above

9. (4 points) Evaluate $\iiint_E y \, dV$, where $E = \{(x, y, z) \mid 0 \le x \le 3,\ 0 \le y \le x,\ x - y \le z \le x + y\}$.
   - A. 9
   - B. $\frac{27}{2}$
   - C. $-\frac{9}{2}$
   - D. $\frac{5}{3}$
   - E. None of the above

10. (4 points) Find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 3t\mathbf{k},\ 0 \le t \le 2,$ if the density is $\delta = 4z$.
    - A. 50
    - B. 115
    - C. 125
    - D. 6π
    - E. None of the above

11. (4 points) Consider the function $f(x, y) = x^2 + xy + y^2$ at the point $(-1, 1)$.
In what direction does f decrease most rapidly?

A. $\frac{1}{\sqrt{2}}\langle 1, 1\rangle$

B. $\frac{1}{\sqrt{2}}\langle 1, -1\rangle$

C. $\frac{1}{\sqrt{2}}\langle -1, 1\rangle$

D. $\frac{1}{\sqrt{2}}\langle -1, -1\rangle$

E. None of the above

12. (4 points) Find the work done by the force $\mathbf{F} = \langle xy, y, -yz\rangle$ over the curve $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t\mathbf{k}$, $0 \leq t \leq 1$ in the direction of increasing t.

A. 1/4

B. 1/3

C. 1/2

D. 1

E. None of the above

13. (4 points) The vector field below could have been generated by which of the following?

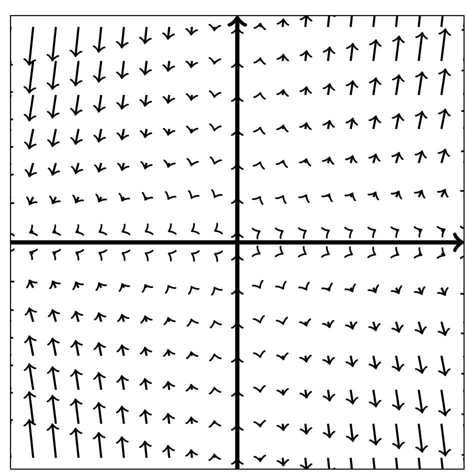
A. $x\mathbf{i} + xy\mathbf{j}$

B. $y\mathbf{i} - x^2\mathbf{j}$

C. $-y\mathbf{i} - xy\mathbf{j}$

D. $x\mathbf{i} - y^2\mathbf{j}$

E. $x^2\mathbf{i} - (y - x)\mathbf{j}$



14. (4 points) Let $\mathbf{F} = \langle 2x+y, x+z, y\rangle$. Which of the following is true?

A. curl $\mathbf{F} = 2x + y$ and div $\mathbf{F} = z - y + x$.

B. curl $\mathbf{F} = \langle 2, 0, 0\rangle$ and div $\mathbf{F} = 0$.

C. curl $\mathbf{F} = \langle 0, 0, 0\rangle$ and div $\mathbf{F} = 2$.

D. curl $\mathbf{F} = \langle z, -y, x+y\rangle$ and div $\mathbf{F} = 2x + y$.

E. None of the above

15. (4 points) Find the absolute maximum and minimum values of $f(x, y) = x^2 - 2x + y^2$ on the set $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

A. $\text{min} = -1,\ \text{max} = 1$

B. $\text{min} = -4,\ \text{max} = 4$

C. $\text{min} = -1,\ \text{max} = 8$

D. $\text{min} = 0,\ \text{max} = 8$

E. None of the above

16. (4 points) Estimate the change of the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ if the point $P(x, y, z)$ moves from $P_0(1, 1, 2)$ a distance of $ds = \frac{1}{5}$ units in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

A. 1

B. $-\frac{1}{\sqrt{5}}$

C. $\frac{1}{\sqrt{5}}$

D. $\frac{1}{21}$

E. None of the above

More Challenging Problem(s). Show all work to receive credit.

17. (6 points) Find the average value of the function $f(x) = \int_x^1 \cos(t^2)\,dt$ on the interval $[0, 1]$.

18. **TRUE or FALSE**? Circle the right choice. No work needed

(a) (2 points) If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}(\mathbf{F}) + \text{curl}(\mathbf{G})$.

A. TRUE
B. FALSE

(b) (2 points) If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\text{curl}(\mathbf{F} \cdot \mathbf{G}) = \text{curl}(\mathbf{F}) \cdot \text{curl}(\mathbf{G})$.

A. TRUE
B. FALSE

(c) (2 points) $\displaystyle\int_{-1}^{1} \int_{0}^{1} e^{x^2+y^2} \sin y\, dx\, dy = 0$

A. TRUE
B. FALSE

(d) (2 points) The integral $\displaystyle\int_0^{2\pi} \int_0^2 \int_r^2 dz\, dr\, d\theta = 0$ represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

A. TRUE
B. FALSE

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. **Please have your MSU student ID ready** so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET PAGE 1

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$

- **Dot Product**:
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\mathbf{u}||\mathbf{v}| \cos\theta$$

- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- **Vector Projection**: $\text{proj}_{\mathbf{u}} \mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)| \, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u}$
- **Second Derivative Test** Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
  (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  (c) If $D < 0$ then $f(a, b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

FORMULA SHEET PAGE 2

Multiple Integrals

- **Area**: $A(D) = \iint_D 1 \; dA$
- **Volume**: $V(E) = \iiint_E 1 \; dV$

Polar/Cylindrical

- Transformations

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$y/x = \tan\theta$$

- $\iint_D f(x,y)\; dA = \iint_D f(r\cos\theta, r\sin\theta)\; r\; dr\; d\theta$
- $\iiint_E f(x,y,z)\; dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\; r\; dz\; dr\; d\theta$

Spherical

- Transformations

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$

- $\iiint_E f(x,y,z)\; dV =$

$$\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2\sin\phi)\; d\rho\; d\phi\; d\theta$$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- $\mathbf{F}$ is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem**

$$\int_C P\; dx + Q\; dy = \iint_D (Q_x - P_y)\, dA$$