Name: ______________________ **PID:** ______________________

Section: ______________ **Recitation Instructor**: ______________________

DO NOT WRITE BELOW THIS LINE. **GO ON TO THE NEXT PAGE.**

Page	Problem	Score	Max Score
3	1		5
	2		5
4	3a		5
	3b		5
	4		5
	5a		5
	5b		5
5	6		5
	7a		5
	7b		5
6	8		18
7	9a		10
	9b		10
8	10a		14
	10b		14
9	10c		14
	10d		14
10	11a		14
	11b		14
11	12a		4
	12b		10
12	13a		4
	13b		10
Total Score			200

Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything excepts pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 12.
- Fill in your name, etc. on the first page and on this page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- You will be given exactly 120 minutes for this exam.
- If you have any questions please raise your hand and a proctor will come to you.

I have read and understand the above instructions: ______________________

SIGNATURE

Score:

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) What is the coefficient next to the x term of the binomial series for the function $f(x) = \left(1 - \frac{x}{3}\right)^{-3}$?

   A. 1
   B. -1
   C. 3
   D. -3
   E. None of the above

2. (5 points) When evaluating the integral $\int \frac{x^3\ dx}{\sqrt{x^2+36}}$ which of the following would be the best substitution for x?

   A. $x = 36 \sin u$
   B. $x = 6 \sin u$
   C. $x = 6 \cos u$
   D. $x = 6 \tan u$
   E. $x = 6 \sec u$

Extra Work Space.

Fill in the Blanks. No work needed. Only possible scores given are 0, 3, and 5.

3. Find the derivatives of the functions defined below.

   (a) (5 points) If $f(x) = 2^x$, the derivative is: $f'(x) \;=\; \underline{\ln(2)2^x}$.

   (b) (5 points) If $g(x) = \left(1+x^2\right)^x$, the derivative is: $g'(x) \;=\; \underline{(1+x^2)^x\left[\ln(1+x^2) + \dfrac{2x^2}{1+x^2}\right]}$.

4. (5 points) Evaluate the improper integral: $\displaystyle\int_{-2}^{1} \frac{1}{x^{4/3}}\,dx \;=\; \underline{\infty \qquad\qquad}$.

5. Consider the function defined by the series: $f(x) = \displaystyle\sum_{n=0}^{\infty} (\sin(x))^n$.

   (a) (5 points) Evaluate $f(x)$ as a simple formula with no summation, assuming the series converges.

   $$f(x) \;=\; \underline{\frac{1}{1-\sin(x)}}$$

   (b) (5 points) For which x does the series converge? $\underline{\text{All } x \neq (2k+1)\pi/2 \text{ for integer } k.}$

Extra Work Space.

6. (5 points) State the general formula for the derivative of an inverse function:
   $(f^{-1})'(x) \;=\; \underline{\dfrac{1}{f'(f^{-1}(x))}}.$

7. Write the Taylor series centered at $x = 0$ for the following functions, in $\sum$-notation.

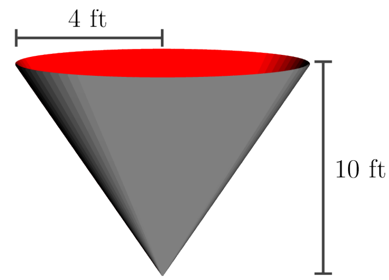
   (a) (5 points) $f(x) \;=\; \dfrac{x^3}{1+x} \;=\; \underline{\sum_{n=0}^{\infty}(-1)^n x^{n+3}}$

   (b) (5 points) $f(x) \;=\; x\cos(x) \;=\; \underline{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}}$

Extra Work Space.

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

8. (18 points) A conical water tank with a top radius of 4 feet and height of 10 feet is standing at ground level as shown in the sketch below. Water weighing 60 pounds per cubic foot is pumped from the tank to an outlet 3 feet above the top of the tank. If the tank is full, how many foot-pounds of work are required to pump all of the water from the tank?



Solution:

$$
\begin{aligned}
Work &= \int_0^{10} 60(13-y)A(y)\ dy \\
&= \int_0^{10} 60(13-y)\pi r^2(y)\ dy \\
&= 60\pi \int_0^{10} (13-y)\left(\frac{2}{5}y\right)^2\ dy \\
&= \frac{48\pi}{5}\int_0^{10}(13-y)y^2\ dy \\
&= \frac{48\pi}{5}\left[\frac{13y^3}{3}-\frac{y^4}{4}\right]_0^{10} \\
&= \boxed{\frac{48\pi}{5}\left[\frac{13000}{3}-\frac{10000}{4}\right]}
\end{aligned}
$$

9. Evaluate the following limits.

(a) (10 points) $\lim_{x\to 0}(1+x)^{1/x} =$

Solution:

$$\begin{aligned}\lim_{x\to 0}(1+x)^{1/x} &= \lim_{x\to 0} e^{\ln(1+x)/x} \\ &= e^{\lim_{x\to 0}\ln(1+x)/x} \\ &= e^{\lim_{x\to 0}(1/(1+x))/1} \\ &= e^1 = \boxed{e}\end{aligned}$$

(b) (10 points) $\lim_{x\to\infty} \frac{\sin(x^2)}{1+\sqrt{x}} =$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{-1}{1+\sqrt{x}} &\le \lim_{x\to\infty}\frac{\sin(x^2)}{1+\sqrt{x}} \le \lim_{x\to\infty}\frac{1}{1+\sqrt{x}} \\ 0 &\le \lim_{x\to\infty}\frac{\sin(x^2)}{1+\sqrt{x}} \le 0\end{aligned}$$

So therefore by Squeeze Thm $\lim_{x\to\infty}\frac{\sin(x^2)}{1+\sqrt{x}} = \boxed{0}$

Notice this does not satisfy the hypotheses for L'Hopitals.

10. Evaluate the following antiderivatives.

(a) (14 points) $\int x \sin(3x)\, dx \ =$

Solution: $u = x \implies du = dx$
$dv = \sin(3x)dx \implies v = \dfrac{-1}{3}\cos(3x)$

$$\begin{aligned}\int x \sin(3x)\, dx &= \frac{-x}{3}\cos(3x) - \int \frac{-1}{3}\cos(3x)dx \\ &= \boxed{\frac{-x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C}\end{aligned}$$

(b) (14 points) $\int \dfrac{4x+6}{x^2+2x-3}\, dx \ =$

Solution:

$$\begin{aligned}\int \frac{4x+6}{x^2+2x-3}\, dx &= \int \frac{4x+6}{(x+3)(x-1)}\, dx \\ &= \int \frac{A}{x+3} + \frac{B}{x-1}\, dx \\ &= A\ln|x+3| + B\ln|x-1| + C\end{aligned}$$

Solving for A and B we get:

$$\begin{aligned}\frac{4x+6}{(x+3)(x-1)} &= \frac{A}{x+3} + \frac{B}{x-1} \\ 4x+6 &= A(x-1) + B(x+3) \\ 4x+6 &= (A+B)x + (3B-A)\end{aligned}$$

Giving $A = 3/2$ and $B = 5/2$ for the final solution $\boxed{\frac{3}{2}\ln|x+3| + \frac{5}{2}\ln|x-1| + C}$

(c) (14 points) $\displaystyle\int \frac{1}{\sqrt{1-2x^2}}\,dx =$

Solution:

$$\begin{aligned}
\int \frac{1}{\sqrt{1-2x^2}}\,dx &= \int \frac{1}{\sqrt{1-u^2}}\,dx && (u=\sqrt{2}x)\\
&= \int \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{\sqrt{2}}\right) && (du=\sqrt{2}\ dx)\\
&= \frac{1}{\sqrt{2}}\arcsin(u)+C\\
&= \boxed{\frac{1}{\sqrt{2}}\arcsin(\sqrt{2}x)+C}
\end{aligned}$$

(d) (14 points) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx =$

Solution:

$$\begin{aligned}
\int \sin^2(x)\cos^3(x)\,dx &= \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx\\
&= \int (\sin^2(x)-\sin^4(x))\cos(x)\,dx\\
&= \int (u^2-u^4)\ du && (u=\sin x, du=\cos x\ dx)\\
&= \left[\frac{u^3}{3}-\frac{u^5}{5}\right]+C\\
&= \boxed{\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+C}
\end{aligned}$$

11. Circle whether the following series converge or diverge, and explain.

(a) (14 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+\sin(n)}$ converges / **diverges** because of Comparison Test.

The Test applies because: a_n positive.

Computations to apply the Test:

Solution:

$$\begin{aligned} \sin n &\leq 1 \\ n + \sin n &\leq n+1 \\ \frac{1}{n+1} &\leq \frac{1}{n+\sin n} \end{aligned}$$

Since $1/(n+1)$ diverges (p-series) we know $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+\sin(n)}$ diverges.

Also can use Limit Comparison with $\frac{1}{n}$.

(b) (14 points) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ **converges** / diverges because of Integral Test

The Test applies because: $a_n = f(n)$ with $f(x)$ positive, continuous, decreasing.

Computations to apply the Test:

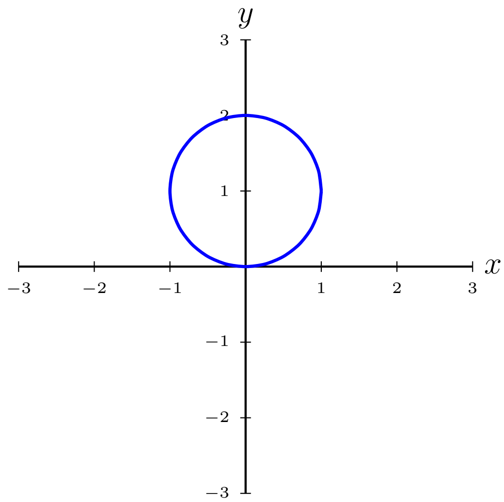
Solution:

$$\begin{aligned} \int_1^{\infty} xe^{-x^2}\,dx &= -\frac{1}{2}e^{-x^2}\big|_{x=1}^{x=N} \\ &= -\frac{1}{2}e^{-N^2} + \frac{1}{2}e^{-1} \to \frac{1}{2e} < \infty \end{aligned}$$

12. Consider the parametric curve given by: $x(t) = \cos(t)$, $y(t) = 1 + \sin(t)$ for $t \in [0, 2\pi]$.

(a) (4 points) Give a sketch of this curve on the axes below which shows all of its important features.

Solution: A circle of radius 1 centered at $(0, 1)$.



(b) (10 points) Give a parametric formula for the tangent line of this curve at $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$.

Solution: $(x(t), y(t)) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ when $t = \pi/6$

$(x'(\pi/6), y'(\pi/6)) = (-\sin(\pi/6), \cos(\pi/6)) = (-1/2, \sqrt{3}/2)$ giving us the solution:

$$(x, y) = \left(\frac{\sqrt{3} - t}{2}, \frac{\sqrt{3}t + 3}{2}\right) \qquad (t \in (-\infty, \infty))$$

13. Consider the curve in polar coordinates: $r = 2\sec(\theta)$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(a) (4 points) Give an equation for the curve in Cartesian (x, y)-coordinates.

Solution:

$$\begin{aligned} r &= 2\sec(\theta) \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

(b) (10 points) **Setup an integral** in polar coordinates that represents the area of the region bounded by this curve and the lines $\theta = 0$ and $\theta = \frac{\pi}{4}$. **Do not evaluate**.

Solution:

$$\begin{aligned} A &= \int \frac{1}{2} r^2 \, d\theta \\ &= \boxed{\int_0^{\frac{\pi}{4}} 2\sec^2(\theta) \, d\theta} \end{aligned}$$

Congratulations you are now done with the exam!

Go back and check your solutions for accuracy and clarity. Make sure your final answers are $\boxed{\text{BOXED}}$.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that it can be checked.