Fall 2014 **MTH 234 FINAL EXAM** December 8, 2014

MTH 234 FINAL EXAM

Name: ______________________________ **PID:** ____________________

Section: ______________ **Instructor**: ______________________________

DO NOT WRITE BELOW THIS LINE. Go to the next page.

Page	Problem	Score	Max Score
1	1		5
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	9		5
	10		5
3	11		10
	12a		8
	12b		4
4	13		16
5	14a		4
	14b		10
	14c		6
6	15a		8
	15b		8
7	16a		6
	16b		6
	17		12
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9	20		12
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Total Score			200

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READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything excepts pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 10.
- Fill in your name, etc. on the first page and on this page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- You will be given exactly 120 minutes for this exam.

I have read and understand the above instructions: ______________________

SIGNATURE

SCORE: ☐

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. A parametric equation for the line through $(1,2,3)$ and perpendicular to the plane $z-x-y=5$ is given by:
   (a) $\mathbf{r}(t)=\langle t-1, 2t-1, 3t+1\rangle$
   (b) $\mathbf{r}(t)=\langle -1-t, -2-t, -t+3\rangle$
   (c) $\mathbf{r}(t)=\langle -t-1, -2t-1, -3t+1\rangle$
   (d) $\mathbf{r}(t)=\langle 1-t, 2-t, 3+t\rangle \longleftarrow$ *Answer*
   (e) None of the above

2. Suppose $x^2+3xy=7$. Which of the following is true?
   (a) $\dfrac{dy}{dx}=\dfrac{3x}{2x+3y}$
   (b) $\dfrac{dy}{dx}=\dfrac{2x+3y}{3x}$
   (c) $\dfrac{dy}{dx}=-\dfrac{3y}{2x+3y}$
   (d) $\dfrac{dy}{dx}=-\dfrac{2x+3y}{3x} \longleftarrow$ *Answer*
   (e) None of the above

3. Let $f=x^2+3xy$ where $x=-v+\sin u$ and $y=u+\sin v$. Which of the following is true?
   (a) $\dfrac{\partial f}{\partial u}=(3u-2v+5\sin v)(\cos u)+(-3v+3\sin u)$
   (b) $\dfrac{\partial f}{\partial u}=(3u-2v+2\sin u+3\sin v)(\cos u)+(-3v+3\sin u) \longleftarrow$ *Answer*
   (c) $\dfrac{\partial f}{\partial u}=(-3u+2v-2\sin u-3\sin v)+(-3v+3\sin u)(\cos v)$
   (d) $\dfrac{\partial f}{\partial u}=2(-v+\sin u)+3(u+\sin v)$
   (e) None of the above

4. The directional derivative of $f(x,y)=x^2(y-1)-2y^2$ at the point $(2,0)$ in the direction of the vector $\mathbf{i}-2\mathbf{j}$ is given by:
   (a) $\big\langle 2x(y-1), x^2-4y\big\rangle \cdot \dfrac{\langle 1,-2\rangle}{\sqrt{5}}$
   (b) $\big\langle 2x(y-1), x^2-4y\big\rangle \cdot \dfrac{\langle 2,0\rangle}{\sqrt{4}}$
   (c) $\langle -4,4\rangle \cdot \dfrac{\langle 1,-2\rangle}{\sqrt{5}} \longleftarrow$ *Answer*
   (d) $\langle -6,9\rangle \cdot \dfrac{\langle 2,0\rangle}{\sqrt{4}}$
   (e) None of the above

Fill in the Blanks. No work needed. Only possible scores given are 0, 3, and 5.

5. Evaluate the limit if it exists. Write DNE if the limit does not exist. $\lim_{(x,y)\to(1,2)} \frac{4x^2-y^2}{2x-y} = \underline{4}$

6. Evaluate the limit if it exists. Write DNE if the limit does not exist. $\lim_{(x,y)\to(0,0)} \frac{4x^2+y^2}{2x^2-y^2} = \underline{\text{DNE}}$

7. If $\frac{d\mathbf{r}}{dt} = (3t^2+1)\mathbf{i} + (4t^3+1)\mathbf{j} - \mathbf{k}$ and $\mathbf{r}(1) = -3\mathbf{i} + 2\mathbf{k}$ then $\mathbf{r}(t) = \underline{\langle t^3+t-5, t^4+t-2, -t+3\rangle}$

8. The integral $\underline{\int_0^1 \sqrt{4t^2+e^{2t}+(\pi\cos(\pi t))^2}\, dt}$ expresses the length of the curve $\mathbf{r}(t) = \langle 1+t^2, e^t, \sin(\pi t)\rangle$ between the points $(1,1,0)$ and $(2,e,0)$. (**Do not evaluate**)

9. $\underline{-4(x-2)+4(y-0)-1(z+4)}$ is an equation of the tangent plane to the surface $z = x^2(y-1) - 2y^2$ at the point $(2,0,-4)$.

10. If $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 f(x,y,z)\, dz\, dy\, dx = \int_0^3 \int_0^a \int_0^b f(x,y,z)\, dx\, dy\, dz$ then $a = \underline{2}$ and $b = \underline{\sqrt{4-y^2}}$.

Extra Work Space.

Standard Response Questions. Show all work to receive credit. Please put your final answer in the $\boxed{\textbf{BOX}}$.

11. (10 points) Find where the lines $L_1(t) = \langle 4+t, 5+t, 6+t\rangle$ and $L_2(s) = \langle 3-s, 6-2s, 1+s\rangle$ intersect

Solution. Solve the system of equations:

$$\begin{aligned} 5+t &= 6-2s \\ 4+t &= 3-s \end{aligned}$$

To give us $\boxed{s=2}$ and $\boxed{t=-3}$. Plugging $t=-3$ into $L_1(t)$ we get

$$\begin{aligned} L_1(-3) &= \langle 4-3, 5-3, 6-3\rangle \\ &= \langle 1,2,3\rangle \end{aligned}$$

Giving us $\boxed{x=1}$, $\boxed{y=2}$, and $\boxed{z=3}$.

12. (8+4=12 points) Given the points $A(1,1,1)$, $B(2,1,0)$, and $C(0,2,3)$.

(a) Find an equation of a plane that through the points A, B, and C.

Solution. Consider the vectors $\overrightarrow{AB} = \langle 1,0,-1\rangle$ and $\overrightarrow{AC} = \langle -1,1,2\rangle$. Then

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{vmatrix} \\ &= (0+1)\mathbf{i} - (2-1)\mathbf{j} + (1-0)\mathbf{k} \\ &= \langle 1,-1,1\rangle \end{aligned}$$

So we get $(x-1)-(y-1)+(z-1)=0 \implies \boxed{z = 1-x+y}$

(b) Find the area of triangle ABC.

Solution. Using the work we did in (a) we know

$$\begin{aligned} \text{Area of triangle ABC} &= \frac{1}{2}|\langle 1,-1,1\rangle| \\ &= \frac{1}{2}\sqrt{3} = \boxed{\frac{\sqrt{3}}{2}} \end{aligned}$$

13. (16 points) Find and classify (max, min or saddle point) the critical points of $f(x, y) = 1 + 6x^2 + 6y^2 - 3x^2y - y^3$. (Hint: There are four.)

Solution. Taking partial derivatives we get $f_x = 12x - 6xy = 6x(2-y)$ and $f_y = 12y - 3x^2 - 3y^2 = -3(y^2 - 4y + x^2)$. Since these are polynomials they are never undefined so we get critical points when $0 = 6x(2-y)$ and $0 = y^2 - 4y + x^2$

$$\begin{array}{ccccc} f_x = 0 & \Longrightarrow & x = 0 & \Longrightarrow & y = 0 \\ & & & \text{or} & y = 4 \\ & \text{or} & y = 2 & \Longrightarrow & x = 2 \\ & & & \text{or} & x = -2 \end{array}$$

giving us the critical points $(0, 0), (0, 4), (2, 2), (-2, 2)$.
Now we calculate second derivatives to classify them:

$$f_{xx} = 12 - 6y \qquad f_{yy} = 12 - 6y \qquad f_{xy} = -6x$$

So we get:

$$D(0,0) = (12)(12) - (0)^2 = 144 > 0$$
$$f_{xx}(0,0) = 12 \qquad \Longrightarrow \boxed{(0,0) \text{ is a local min}}$$

$$D(0,4) = (-12)(-12) - (0)^2 = 144 > 0$$
$$f_{xx}(0,4) = -12 \qquad \Longrightarrow \boxed{(0,4) \text{ is a local max}}$$

$$D(2,2) = (0)(0) - (-12)^2 = -144 < 0 \qquad \Longrightarrow \boxed{(2,2) \text{ is a saddle pt}}$$

$$D(-2,2) = (0)(0) - (12)^2 = -144 < 0 \qquad \Longrightarrow \boxed{(2,-2) \text{ is a saddle pt}}$$

14. (4+10+6=20 points) Let $\mathbf{F} = \langle 2x + y + yz, x + xz, xy + 1 \rangle$.

(a) Show that $\text{curl}(\mathbf{F}) = \mathbf{0}$.

Solution.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+y+yz & x+xz & xy+1 \end{vmatrix} = (x-x)\mathbf{i} - (y-y)\mathbf{j} + ((1+z)-(1+z))\mathbf{k}$$
$$= \langle 0, 0, 0 \rangle$$

(b) Find a potential function f such that $\nabla f = \mathbf{F}$.

Solution.

$$\begin{aligned}
\int f_x \, dx &= \int 2x + y + yz \, dx \\
f &= x^2 + xy + xyz + g(y,z) && \text{(integrate } (\star)) \\
f_y &= x + xz + g_y(y,z) && \text{(take partial derivative)} \\
x + xz &= x + xz + g_y(y,z) && \text{(Plug in } Q \text{ for } f_y) \\
0 &= g_y(y,z) && \text{(algebra)} \\
g(z) &= g(y,z) && \text{(integrate)} \\
f &= x^2 + xy + xyz + g(z) && \text{(substitute into } (\star)) \\
f_z &= xy + g_z(z) && \text{(take partial derivative)} \\
xy + 1 &= xy + g_z(z) && \text{(Plug in } R \text{ for } f_z) \\
1 &= g_z(z) && \text{(Algebra)} \\
z + K &= g(z) && \text{(integrate)} \\
f &= \boxed{x^2 + xy + xyz + z + K} && \text{(substitute into } (\star))
\end{aligned}$$

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a curve from the point $(0,0,0)$ to the point $(2,1,1)$.

Solution. $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,1,1) - f(0,0,0) = (4+2+2+1) - (0) = \boxed{9}$.

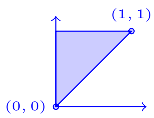
15. (8+8=16 points) Evaluate the integrals

(a) $\displaystyle\int_0^1 \int_x^1 4e^{(x/y)}\, dy\, dx.$

Solution. This can not be so easily integrated. Lets switch the limits of integration.

$$\begin{aligned}
\int_0^1 \int_x^1 4e^{(x/y)}\, dy\, dx &= \int_0^1 \int_0^y 4e^{(x/y)}\, dx\, dy \\
&= \int_0^1 \left[4ye^{(x/y)}\right]_0^y\, dy \\
&= \int_0^1 \left[4ye^1 - 4y\right]\, dy \\
&= \int_0^1 4y(e-1)\, dy \\
&= \left[2y^2(e-1)\right]_0^1 = \boxed{2(e-1)}
\end{aligned}$$

Sketch a mini picture

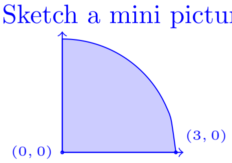


(b) $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} 2\cos(x^2+y^2)\ dy\ dx.$

Solution. Polar coordinates for the win.

$$\begin{aligned}
\int_0^3 \int_0^{\sqrt{9-x^2}} 2\cos(x^2+y^2)\ dy\ dx &= \int_0^{\pi/2} \int_0^3 2\cos(r^2)\ r\ dr\ d\theta \\
&= \frac{\pi}{2} \int_0^3 2r\cos(r^2)\ dr \\
&= \frac{\pi}{2} \left[\sin(r^2)\right]_0^3 \\
&= \boxed{\frac{\pi}{2}\sin(9)}
\end{aligned}$$

Sketch a mini picture



16. (6+6=12 points)

(a) Express $(x-2)^2+y^2=4$ in terms of polar coordinates. Solve for r. Simplify as much as possible.

Solution.

$$\begin{aligned} (r\cos\theta-2)^2+(r\sin\theta)^2 &= 4 \\ r^2\cos^2\theta-4r\cos\theta+4+r^2\sin^2\theta &= 4 \\ r^2-4r\cos\theta &= 0 \\ r(r-4\cos\theta) &= 0 \\ r &= \boxed{4\cos\theta} \qquad (r=0 \text{ is only a dot, not a circle. Extraneous.}) \end{aligned}$$

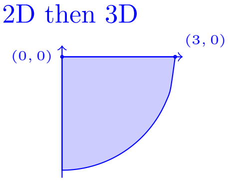
(b) Express (DO NOT EVALUATE) a triple integral in cylindrical coordinates for the volume of the portion of the sphere $x^2+y^2+z^2=16$ contained within the cylinder $(x-2)^2+y^2=4$.

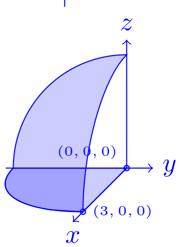
Solution. The surface we enter our region of integration through is $z=-\sqrt{16-x^2-y^2}=-\sqrt{16-r^2}$ and we exit through $z=\sqrt{16-x^2-y^2}=\sqrt{16-r^2}$. Finally we should note that a possible domain of $r=4\cos\theta$ to cover the circle once is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. So we get.

$$\begin{aligned} \iiint_E 1\ dV &= \int\int\int r\ dz\ dr\ d\theta \\ &= \boxed{\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\ dz\ dr\ d\theta} \end{aligned}$$

17. (12 points) Evaluate the integral $\displaystyle\int_{-3}^{0}\int_0^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2-y^2}}(4z)\ dz\,dx\,dy$ in spherical coordinates.

Solution.
Sketch a mini picture in 2D then 3D





These should suffice to give us:

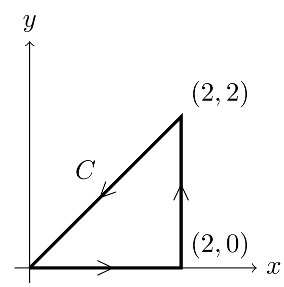
$$\begin{aligned} \int_{-3}^{0}\int_0^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2-y^2}}(4z)\ dz\,dx\,dy &= \iiint(4\rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta \\ &= \int_{-\pi/2}^{0}\int_0^{\pi/2}\int_0^3 4\rho^3\sin\phi\cos\phi\ d\rho\ d\phi\ d\theta \\ &= \frac{\pi}{2}\int_0^{\pi/2}\left[\rho^4\sin\phi\cos\phi\right]_0^3\ d\phi \\ &= \frac{\pi}{2}\int_0^{\pi/2}\left[81\sin\phi\cos\phi\right]\ d\phi \\ &= \frac{\pi}{4}\left[81\sin^2\phi\right]_0^{\pi/2} \\ &= \boxed{\frac{\pi}{4}[81]} \end{aligned}$$

18. (12 points) Find the work done by the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$ over the path $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \le t \le \dfrac{\pi}{2}$.

Solution.

$$\begin{aligned}
\int_0^{\pi/2} \left\langle y, -x, z^2 \right\rangle \cdot d\mathbf{r} &= \int_0^{\pi/2} \left\langle y, -x, z^2 \right\rangle \cdot \langle -\sin t, \cos t, 1 \rangle \ dt \\
&= \int_0^{\pi/2} \left\langle \sin t, -\cos t, t^2 \right\rangle \cdot \langle -\sin t, \cos t, 1 \rangle \ dt \\
&= \int_0^{\pi/2} -\sin^2 t - \cos^2 t + t^2 \ dt \\
&= \int_0^{\pi/2} t^2 - 1 \ dt \\
&= \left[\frac{t^3}{3} - t \right]_0^{\pi/2} = \boxed{\frac{\pi^3}{24} - \frac{\pi}{2}}
\end{aligned}$$

19. (12 points) Use Green's Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \left\langle 6y + \frac{\sin^2 x}{10+x^4}, 3y^9\cos(y^2) - 4x \right\rangle$ and C is the positively oriented triangle shown below.



Solution.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \left\langle 6y + \frac{\sin^2 x}{10 + x^4}, 3y^9\cos(y^2) - 4x \right\rangle \cdot d\mathbf{r} \\
&= \iint_D (-4 - 6) \ dA \\
&= -10 \iint_D \ dA \\
&= -10 \left[\frac{1}{2}(2)(2) \right] = \boxed{-20}
\end{aligned}$$

20. (12 points) Find the surface area of the paraboloid $z = 5 - 2x^2 - 2y^2$ that lies above the plane $z = -13$.

Solution. These words give us $f(x, y) = 5 - 2x^2 - 2y^2$ and

$$\begin{aligned} 5 - 2x^2 - 2y^2 &\geq -13 \\ 9 &\geq x^2 + y^2 \end{aligned}$$

Giving us $D = \{(x, y) \mid 9 \geq x^2 + y^2\}$. Using the surface area formula we get:

$$\begin{aligned} \iint_D \sqrt{1 + [f_x]^2 + [f_y]^2}\, dA &= \iint_D \sqrt{1 + 16x^2 + 16y^2}\, dA \\ &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 16r^2}\, r\, dr\, d\theta \\ &= 2\pi \int_0^3 \frac{32r}{32} \sqrt{1 + 16r^2}\, dr \\ &= 2\pi \left[\frac{2}{3(32)} (1 + 16r^2)^{3/2} \right]_0^3 \\ &= \frac{4\pi}{96} \left[(1 + 16(9))^{3/2} - 1 \right] = \boxed{\frac{\pi}{24} \left[(145)^{3/2} - 1 \right]} \end{aligned}$$

21. (16 points). Let $\mathbf{F} = (xy^2 - z)\mathbf{i} + (12x + yz^2)\mathbf{j} + (zx^2 - \sin x)\mathbf{k}$ and let S be the sphere $x^2 + y^2 + z^2 = 4$. Use the Divergence Theorem to find evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution.

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div } \mathbf{F}\, dV \\ &= \iiint_E (y^2 + z^2 + x^2)\, dV \\ &= \iiint_E (\rho^2) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^2 (\rho^2) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= 2\pi \int_0^{\pi} \left[\frac{\rho^5}{5} \sin\phi \right]_0^2 d\phi \\ &= \frac{64\pi}{5} \left[-\cos\phi \right]_0^{\pi} = \boxed{\frac{128\pi}{5}} \end{aligned}$$