

Name: _____

Section: _____ Recitation Instructor: _____

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 120 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the
above instructions and statements
regarding academic honesty: _____

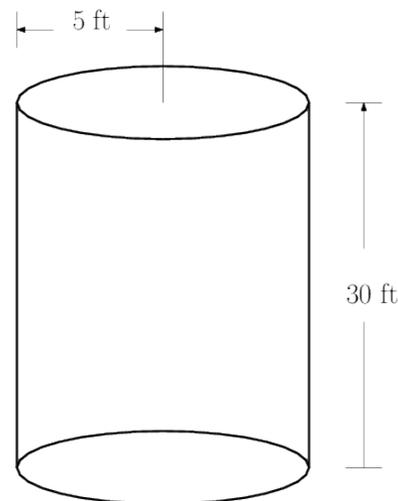
SIGNATUREScore: /108

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (6 points) How much work does it take to pump the water from a full upright circular cylindrical tank of radius 5 ft and height 30 ft to a level of 10 ft above the top of the tank? (Water weighs 63 lbs/ft³.)

Solution:

$$\begin{aligned} W &= \int_0^{30} 63 \cdot (40 - y)(\pi 5^2) dy \\ &= 63 \cdot 25\pi \left[40y - \frac{y^2}{2} \right]_0^{30} \\ &= 63 \cdot 25\pi \left[40(30) - \frac{900}{2} \right] = 63 \cdot 25\pi \cdot 750 \text{ ft} \cdot \text{lbs} \end{aligned}$$



2. (6 points) Solve the initial value problem $\frac{dy}{dx} = \frac{y}{1+x^2}$ with $y(0) = 2$.

Solution:

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{1+x^2} \\ \ln y &= \arctan(x) + C \\ y &= Ke^{\arctan(x)} \\ 2 &= Ke^0 \\ y &= 2e^{\arctan(x)} \end{aligned}$$

3. Determine whether each of the series converges or diverges. Show your work and justify your use of any convergence tests.

(a) (6 points) $\sum_{n=1}^{\infty} \frac{3}{n^2 - 2n + 2}$

Solution: Consider comparing with the series $\frac{1}{n^2}$.

$$\lim_{n \rightarrow \infty} \frac{\frac{3}{n^2 - 2n + 2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{3n^2}{n^2 - 2n + 2} = 3$$

Therefore series converges by the limit comparison test and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -series test.

(b) (6 points) $\sum_{n=1}^{\infty} \frac{2^n n^3}{(n+1)!}$

Solution: Consider the ratio test

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}(n+1)^3}{(n+2)!}}{\frac{2^n n^3}{(n+1)!}} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n+1}(n+1)^3}{(n+2)!} \cdot \frac{(n+1)!}{2^n n^3} \\ &= \lim_{n \rightarrow \infty} \frac{2(n+1)^3}{(n+2)} \cdot \frac{1}{n^3} = 0 < 1 \end{aligned}$$

So the series converges by the Ratio test

4. (6 points) Evaluate the definite integral $\int_0^1 xe^x dx$.

Solution: Consider integration by parts with $u = x$, $dv = e^x dx$, $du = dx$ and $v = e^x$.

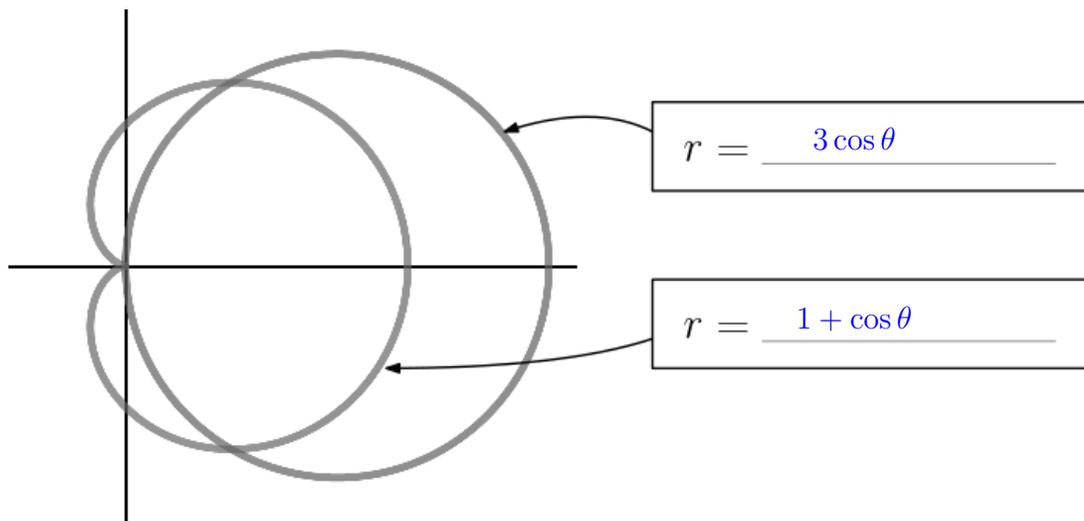
$$\begin{aligned}\int_0^1 xe^x dx &= \left[x \cdot e^x \right]_0^1 - \int_0^1 e^x dx \\ &= [e] - (e^1 - e^0) = 1\end{aligned}$$

5. (6 points) Find the Maclaurin series of the function $f(x) = \cos(x^4)$.

Solution:

$$\begin{aligned}\cos(x^4) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n}}{(2n)!}\end{aligned}$$

6. (a) (3 points) Shown on the plot below are the two polar curves $r = 3 \cos \theta$ and $r = 1 + \cos \theta$. Label the plot indicating which curve is which.



- (b) (5 points) Find all points of intersections between the two curves in part (a). Express your answer in polar coordinates (r, θ) .

Solution: From the picture we can see they intersect at the origin, that is, $(r, \theta) = (0, \star)$ is an intersection point where \star can be any value. Also the curves intersect at

$$\begin{aligned} 3 \cos \theta &= 1 + \cos \theta \\ \cos \theta &= 1/2 \\ \theta &= \pi/3, 5\pi/3 \end{aligned}$$

Which is at the points $(r, \theta) = (3/2, \pi/3)$ and $(r, \theta) = (3/2, 5\pi/3)$.

- (c) (4 points) Set up, but do not evaluate, an integral which computes the area **in the first quadrant** that is inside $r = 3 \cos \theta$ and outside $r = 1 + \cos \theta$.

Solution:

$$A = \int_0^{\pi/3} \frac{1}{2} [(3 \cos \theta)^2 - (1 + \cos \theta)^2] d\theta$$

7. Compute the following indefinite integrals.

(a) (6 points) $\int \frac{4x^2 + 3}{x^3 + x} dx$

Solution: $\frac{4x^2 + 3}{x^3 + x} = \frac{4x^2 + 3}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$ which yields $Ax^2 + A + Bx^2 + Cx = 4x^2 + 3$ giving $A = 3$, $B = 1$ and $C = 0$

$$\begin{aligned}\int \frac{4x^2 + 3}{x^3 + x} dx &= \int \frac{3}{x} + \frac{x}{x^2 + 1} dx \\ &= 3 \ln |x| + \frac{1}{2} \ln(x^2 + 1) + C\end{aligned}$$

(b) (6 points) $\int \frac{x^2}{\sqrt{4 - x^2}} dx$

Solution: Use trigonometric substitution with $x = 2 \sin \theta$ and $dx = 2 \cos \theta d\theta$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{4 - x^2}} dx &= \int \frac{4 \sin^2 \theta}{\sqrt{4 - 4 \sin^2 \theta}} (2 \cos \theta d\theta) \\ &= \int 4 \sin^2 \theta d\theta \\ &= \int 2 - 2 \cos(2\theta) d\theta \\ &= 2\theta - \sin(2\theta) \\ &= 2 \sin^{-1} \left(\frac{x}{2} \right) - \frac{x\sqrt{4 - x^2}}{2} + C\end{aligned}$$

Multiple Choice. Circle the best answer. No work needed.

No partial credit available. No credit will be given for choices not clearly marked.

8. (3 points) Let R be the region in the first quadrant bounded by the x -axis, the line $x = 4$, and $y = \sqrt{x}$. The volume of the solid of revolution formed by revolving R about the x -axis is given by:

- A. $\int_0^2 \pi y^2 dy$
- B. $\int_0^2 \pi y^4 dy$
- C. $\int_0^4 \pi \sqrt{x} dx$
- D. $\int_0^4 \pi x^2 dx$
- E.** $\int_0^4 \pi x dx$

9. (3 points) The derivative of $y = (\ln x)(\sin^{-1} x)$ is equal to

- A. $\frac{\sin^{-1} x}{x} + \frac{\ln x}{1+x^2}$
- B.** $\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}}$
- C. $\frac{1}{x} \frac{1}{\sqrt{1-x^2}}$
- D. $\frac{\ln x}{\sqrt{1-x^2}}$
- E. None of the above.

10. (3 points) Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

- A. 0
- B.** $1/2$
- C. 1
- D. 2
- E. The limit does not exist.

11. (3 points) The derivative of the function $y = x^x$ is equal to

- A. $x \cdot x^{x-1}$
- B. $x^x \cdot \ln x$
- C. $(x - 1)^{x-1}$
- D. $x^x \cdot (1 + \ln x)$**
- E. x^x

12. (3 points) Which of the following series converge?

$$(a) \sum_{n=1}^{\infty} \frac{1}{n} \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}} \quad (c) \sum_{n=1}^{\infty} \frac{(-1)^n}{1 - n^{-1}} \quad (d) \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

- A. (b) and (c) only.
- B. (a), (b), and (d) only.
- C. (b) and (d) only.**
- D. (b), (c) and (d) only.
- E. None of the above choices are correct.

13. (3 points) Which of the following series converge absolutely?

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad (b) \sum_{n=1}^{\infty} \frac{1}{2^n} \quad (c) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad (d) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$$

- A. (b) and (c) only.**
- B. (a), (b), and (d) only.
- C. (b) and (d) only.
- D. (b), (c) and (d) only.
- E. None of the above choices are correct.

14. (3 points) Evaluate $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x$.

- A. 0
- B. $1/e$
- C. 1**
- D. e
- E. The limit does not exist.

15. (3 points) The Maclaurin series of the function $\sqrt{1+x}$ starts with the terms:

- A. $1 + \frac{1}{2}x^2 - \frac{1}{4!}x^4$
- B. $1 + \frac{1}{2}x - \frac{1}{8}x^2$**
- C. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3$
- D. $1 + \frac{1}{2}x + \frac{1}{8}x^2$
- E. $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

16. (3 points) The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2 2^n}$ is:

- A. -3
- B. $1/2$
- C. 1
- D. 2**
- E. 5

17. (3 points) Let $f(x) = x + \sqrt{x}$. Find $(f^{-1})'(2)$ given that $f(1) = 2$.

- A. $2/3$
- B. $3/2$
- C. $1 + 1/(2\sqrt{2})$
- D. $2\sqrt{2}/(1 + 2\sqrt{2})$
- E. $2 + \sqrt{2}$

18. (3 points) The length of the curve given in polar coordinates by $r = 1 + \cos \theta$, for $0 \leq \theta \leq 2\pi$, is given by:

- A. $\int_0^{2\pi} \sqrt{1 + \sin^2 \theta} d\theta$
- B. $\int_0^{2\pi} \sqrt{1 + \cos \theta} d\theta$
- C. $\int_0^{2\pi} (1 + \cos \theta)^2 d\theta$
- D. $\int_0^{2\pi} \sqrt{1 + 2 \cos \theta} d\theta$
- E. $\int_0^{2\pi} \sqrt{2 + 2 \cos \theta} d\theta$

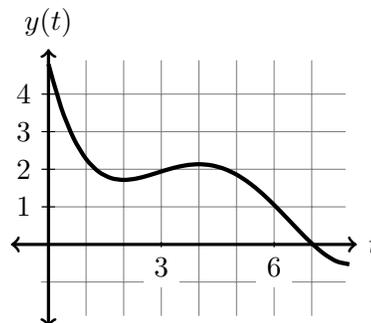
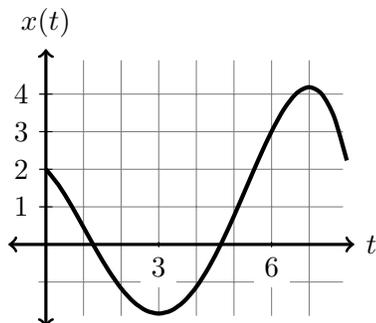
19. (3 points) Which of the following improper integrals converge?

$$(a) \int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (b) \int_0^1 \frac{1}{\sqrt{x}} dx \quad (c) \int_1^{\infty} \frac{1}{x^2} dx \quad (d) \int_0^1 \frac{1}{x^2} dx.$$

- A. (b) and (c) only.**
- B. (a), (b), and (d) only.
- C. (b) and (d) only.
- D. (b), (c) and (d) only.
- E. None of the above choices are correct.

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

20. (6 points) **Approximate** the equation in x and y for the line tangent to the parametric curve $(x(t), y(t))$ at $t = 6$. The graphs of $x(t)$ and $y(t)$ are shown below.



Solution: When $t = 6$ it appears $x(6) = 3$, $x'(6) = 2$, $y(6) = 1$ and $y'(6) = -1$. In which case the tangent line would be

$$y - 1 = -\frac{1}{2}(x - 3)$$

21. (6 points) Find **all** values of x for which the series $\sum_{n=0}^{\infty} \cos^n x$ **diverges**. (Make sure you justify both that the series diverges for the values of x in your list, and that the series converges for all other values.)

Solution: A geometric series $\sum_{n=0}^{\infty} r^n$ converges when $|r| < 1$ and $r = \cos x$ has the range of $[-1, 1]$ so the only values on which the series diverges is when

$$\begin{aligned} \cos(x) &= \pm 1 \\ x &= k\pi \end{aligned} \quad (\text{for all integers } k)$$