Name: ______________________ **PID:** ______________

Section: __________ **Recitation/Instructor**: ______________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 11.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- You will be given exactly 120 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page for scratch work. **Work on the back of pages will not be graded.**
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (4 points) Find an equation of a plane that contains the point $(1, 2, 3)$ and is normal to the line $\mathbf{L}(t) = \langle 3t, 1-t, 0\rangle$.

Solution:

$$\begin{aligned} 3(x-1) - 1(y-2) + 0(z-3) &= 0 \\ 3x - y &= 1 \end{aligned}$$

2. (4 points) Find the distance between the point $(0, 0, 1)$ and the plane $z = 5 + 2x + 3y$.

Solution: A point on the plane is $R = (0, 0, 5)$. If $P = (0, 0, 1)$ then $\overrightarrow{PR} = \langle 0, 0, 4\rangle$ and $\mathbf{n} = \langle 2, 3, -1\rangle$ so

$$\begin{aligned} d &= \frac{|\overrightarrow{PR} \cdot \mathbf{n}|}{|\mathbf{n}|} \\ &= \frac{|\,\langle 0,0,4\rangle \cdot \langle 2,3,-1\rangle\,|}{|\,\langle 2,3,-1\rangle\,|} \\ &= \frac{|-4|}{\sqrt{4+9+1}} \\ &= \frac{4}{\sqrt{14}} \end{aligned}$$

3. (4 points) Find the area of the triangle with vertices $(0, 0, 0)$, $(1, 2, 0)$, and $(3, 0, 5)$.

Solution: Let $A = (0, 0, 0)$, $B = (1, 2, 0)$, and $C(3, 0, 5)$ then $\overrightarrow{AB} = \langle 1, 2, 0\rangle$ and $\overrightarrow{AC} = \langle 3, 0, 5\rangle$. The area of the triangle is given by

$$\begin{aligned} \text{Area } \triangle ABC &= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \\ &= \frac{1}{2}|\,\langle 1,2,0\rangle \times \langle 3,0,5\rangle\,| \\ &= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 3 & 0 & 5 \end{vmatrix} \\ &= \frac{1}{2}|10\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}| = \frac{1}{2}\sqrt{100+25+36} = \frac{\sqrt{161}}{2} \end{aligned}$$

4. (6 points) Show that the following limit does not exist: $\lim_{(x,y)\to(1,0)} \left[\frac{xy}{x+y-1}\right]$

Solution:
Path 1: $x = 1$ and $y \to 0$

$$\lim_{\substack{x=1 \\ y\to 0}} \left[\frac{xy}{x+y-1}\right] = \lim_{y\to 0} \left[\frac{(1)y}{(1)+y-1}\right] = 1$$

Path 2: $y = 0$ and $x \to 1$

$$\lim_{\substack{y=0 \\ x\to 1}} \left[\frac{xy}{x+y-1}\right] = \lim_{x\to 1} \left[\frac{x(0)}{x+(0)-1}\right] = 0$$

Since the limit yields different results along different paths the limit DNE.

5. (6 points) Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \frac{y}{x^5+1}\, dx\, dy$

Solution: To evaluate this we need to switch the order of integration

$$\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 \frac{y}{x^5+1}\, dx\, dy &= \int_0^2 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx \\
&= \int_0^2 \frac{\frac{1}{2}x^4}{x^5+1}\, dx \\
&= \left[\frac{1}{2}\cdot\frac{1}{5}\ln|x^5+1|\right]_0^2 \\
&= \frac{1}{10}\ln(33)
\end{aligned}$$

6. Consider the vector field $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + (5 + xy)\mathbf{k}$

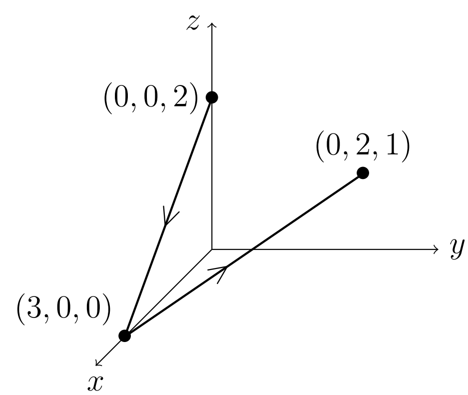
(a) (4 points) Calculate curl($\mathbf{F}$)

Solution:

$$\text{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & 5+xy \end{vmatrix} = (x-x)\mathbf{i} - (y-y)\mathbf{j} + (z-z)\mathbf{k} = \langle 0,0,0 \rangle$$

(b) (8 points) Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$

where C is consists of the line segment from $(0,0,2)$ to $(3,0,0)$, followed by the line segment from $(3,0,0)$ to $(0,2,1)$ shown below.

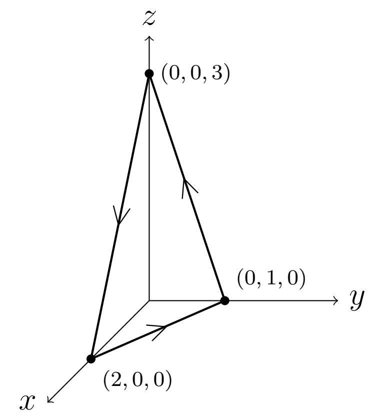


Solution: Consider $f = xyz + 5z$ then $\nabla f = \mathbf{F}$. Using the fundamental theorem of line integrals we know

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,2,1) - f(0,0,2) = (0+5) - (0+10) = -5$$

7. (12 points) C is the triangle with vertices $(2,0,0)$, $(0,1,0)$, and $(0,0,3)$ oriented counterclockwise as viewed from above.

   Use Stokes' Theorem to evaluate $\int_C \left\langle 5x^2, x, z^3 \right\rangle \cdot d\mathbf{r}$



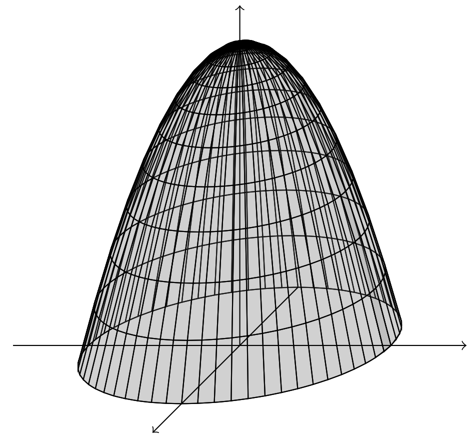
Solution: The equation of the plane that contains these 3 points is $\frac{x}{2} + y + \frac{z}{3} = 1$. Solving for an explicit equation to the surface we get $z = 3 - \frac{3x}{2} - 3y$ so by Stokes' Theorem

$$\begin{aligned}\int_C \left\langle 5x^2, x, z^3 \right\rangle \cdot d\mathbf{r} &= \iint_S \text{curl}(\left\langle 5x^2, x, z^3 \right\rangle) \cdot d\mathbf{S} \\ &= \iint_D \langle 0, 0, 1 \rangle \cdot \pm \left\langle \tfrac{-3}{2}, -3, -1 \right\rangle \; dA \\ &= \iint_D 1 \; dA \\ &= \tfrac{1}{2}(2)(1) = 1\end{aligned}$$

8. (12 points) Let S be the closed surface that consists of the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$ together with the disc in the xy-plane $x^2 + y^2 \leq 4$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = 3x^2\mathbf{i} + 2xy\mathbf{j} + 5z\mathbf{k}$$

Compute the outward flux: $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



Solution: Using the divergence theorem

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (8x + 5)\ dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (8r\cos\theta + 5)r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^2 (8r^2\cos\theta + 5r)(4 - r^2)\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^2 32r^2\cos\theta + 20r - 8r^4\cos\theta - 5r^3\ dr\ d\theta \\
&= \int_0^{2\pi} \left[\frac{32}{3}r^3\cos\theta + 10r^2 - \frac{8}{5}r^5\cos\theta - \frac{5}{4}r^4\right]_0^2\ d\theta \\
&= \int_0^{2\pi} \left[\frac{32}{3}(8)\cos\theta + 10(4) - \frac{8}{5}(32)\cos\theta - \frac{5}{4}(16)\right]\ d\theta \\
&= \left[\frac{32}{3}(8)\sin\theta + 40\theta - \frac{8}{5}(32)\sin\theta - 20\theta\right]_0^{2\pi} \\
&= [40(2\pi) - 20(2\pi)] = 40\pi
\end{aligned}$$

Multiple Choice. Circle the best answer. No work needed.
No partial credit available. No credit will be given for choices not clearly marked.

9. (3 points) Suppose $\mathbf{r}'(t) = 12t\mathbf{i} + 12t^2\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} - \mathbf{k}$. Find $\mathbf{r}(1)$.

A. $7\mathbf{i} + 4\mathbf{j}$
B. $6\mathbf{i} + 4\mathbf{j} + \mathbf{k}$
C. $6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
D. $5\mathbf{i} + \mathbf{k}$
E. None of the above.

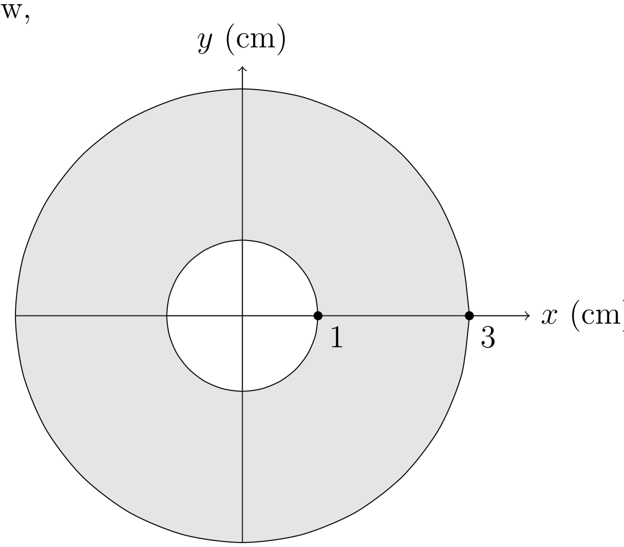
10. (3 points) Find the length of the curve $\mathbf{r}(t) = \langle t^3, 6t, 3t^2 \rangle$, $0 \leq t \leq 1$.

A. 15/2
B. 20
C. 7
D. 13
E. 9

11. (3 points) If z is given implicitly as a function of x and y by the equation $e^z - x^2y - y^2z = 0$ then $\dfrac{\partial z}{\partial y}$ is:

A. $\dfrac{x^2 + 2yz}{e^z - y^2}$
B. $\dfrac{x^2 - 2yz}{e^z}$
C. $\dfrac{x^2}{e^z + y^2}$
D. $\dfrac{x^2 - 2yz}{e^z - y^2}$
E. $\dfrac{1}{e^z - y^2}$

12. (3 points) Calculate the mass of the washer shown below, whose density is given by $\sigma(x, y) = x^2 + y^2$ grams/cm^2.



A. 40π **grams**

B. $52\pi/3$ grams

C. $26\pi/3$ grams

D. 52π grams

E. 80π grams

13. (3 points) What is the range of $f(x, y) = 2 - \sqrt{1 + x + y}$

A. $[0, \infty)$

B. $(-\infty, 2]$

C. $(-\infty, 1]$

D. $[1, \infty)$

E. $[2, \infty)$

14. (3 points) Evaluate the limit $\displaystyle\lim_{(x,y)\to(2,1)} \frac{2y - x}{x^2 - 4y^2}$

A. $-1/4$

B. $1/2$

C. -2

D. 1

E. The limit does not exist.

15. (3 points) Use a linear of approximation of $f(x, y) = 3y + x^2$ to compute an approximate value of $f(0.9, 2.2)$.

   A. 7.1

   B. 7.4

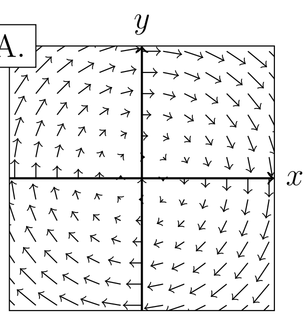
   C. 7.5

   D. 7.3

   E. 7.2

16. (3 points) Which of the following could be a graph of $\mathbf{F} = \langle y, -x \rangle$?

A.



17. (3 points) What is the absolute maximum of $f(x, y) = 2xy - y^2$ on **the boundary** of $D = \{(x, y) \mid x \in [0, 2],\ y \in [0, 3]\}$?

   A. 3

   B. 4

   C. 5

   D. 6

   E. 7

18. (3 points) For which values of a and b is $x^2 + y^2 + az^2 = b$ a hyperboloid of two sheets?

A. $a = 1$ and $b = 1$

B. $a = 1$ and $b = -1$

C. $a = -1$ and $b = 1$

D. $a = -1$ and $b = -1$

E. None of the above.

19. (3 points) Suppose P is a critical point of $f(x, y)$. Under which situation is $f(P)$ a **local maximum**?

A. $f_{xx}(P) = -3$, $f_{xy}(P) = 2$, and $f_{yy}(P) = -2$.

B. $f_{xx}(P) = -1$, $f_{xy}(P) = 1$, and $f_{yy}(P) = -1$.

C. $f_{xx}(P) = 0$, $f_{xy}(P) = 1$, and $f_{yy}(P) = 0$.

D. $f_{xx}(P) = 1$, $f_{xy}(P) = 5$, and $f_{yy}(P) = 1$.

E. $f_{xx}(P) = 3$, $f_{xy}(P) = 2$, and $f_{yy}(P) = 2$.

20. (3 points) Find the volume between the plane $y - x + z = 3$ (shown to the right) and xy-plane over the region D.

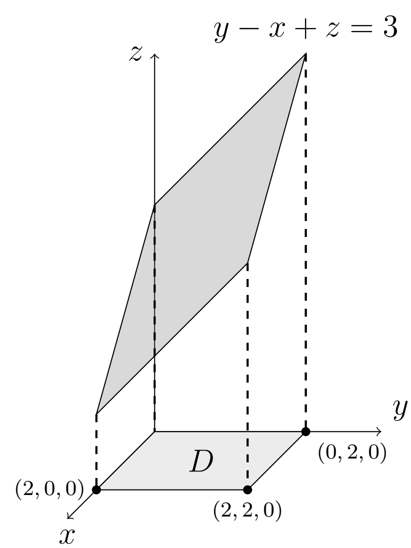
A. 2

B. 3

C. 6

D. 9

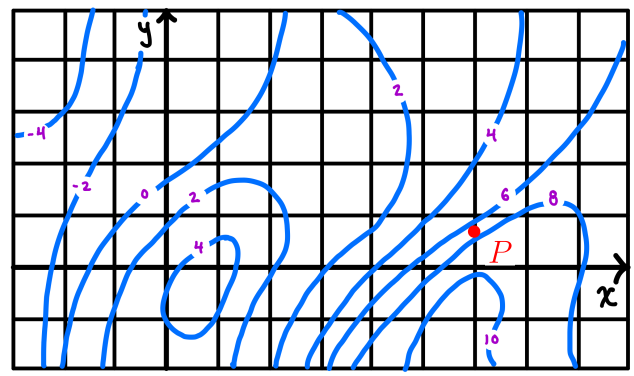
E. 12



Challenging Question(s). Show all work to receive credit.

21. (6 points) Consider the level curves (contour plot) of a function $f(x, y)$ to the right. Plot a point P for which you think $|\nabla f(P)|$ is the greatest. Explain how/why you made your choice.

Solution: $|\nabla f(P)|$ measures the rate of change of f. The contour lines show the values of f so P is placed where the contour lines are closest together, indicating a fast change in the function's value.



22. (6 points) Consider the surface S given by $y = x^2 + z^2$ and bounded by $y = 1$.
Evaluate $\iint_S 1\, dS$ and give the geometric interpretation to your answer.

Solution: There are multiple ways to do this problem... Here is one that is especially clever.

$\iint_S 1\, dS$ is the surface area of the described surface. By interchanging $y \leftrightarrow z$ it does not change the surface's area, it only reflects the surface of the yz-plane.

So instead consider $\tilde{S}$ given by $z = x^2 + y^2$ and bounded by $z = 1$.

$$
\begin{aligned}
A(S) = A(\tilde{S}) &= \iint \sqrt{(2x)^2 + (2y)^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \cdot r\, dr\, d\theta \\
&= 2\pi \left[\frac{1}{12}(5\sqrt{5} - 1) \right] = \frac{\pi}{6}(5\sqrt{5} - 1)
\end{aligned}
$$