

MATH 106

Lecture 2

Permutations & Combinations

© m j winter FS2004

1



Player Numbers

Shirts have 2-digit numbers

Six possible digits: 0, 1, 2, 3, 4, 5

How many different numbers?

$$\boxed{6} \boxed{6} = 36$$

6 possibilities for first digit; 6 for second



Suppose double digits are not allowed? You have one of each the numbers; you select two and iron them on. How many choices have you for one shirt?

$$6 * 5 = 30$$

2

Permutations - Order Matters

The number of ways one can select 2 items from a set of 6, with order mattering, is called

the **number of permutations of 2 items selected from 6**

$$6 \times 5 = 30 = {}_6P_2$$

Example: The final night of the Folklore Festival will feature 3 different bands. There are 7 bands to choose from. How many different **programs** are possible?

3

Calculating ${}_nP_r$

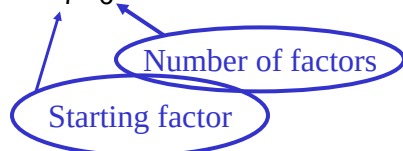
Solution to band problem: ${}_7P_3 = 7 \cdot 6 \cdot 5 = 210$

This is not the same as asking "How many ways are there to choose 3 bands from 7?"

Write out expressions for

$${}_{52}P_4 = 52 \cdot 51 \cdot 50 \cdot 49$$

$${}_7P_6 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$$



4

Factorial Notation for nPr

$${}_7P_3 = 7 \cdot 6 \cdot 5 = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$${}_{52}P_4 = 52 \cdot 51 \cdot 50 \cdot 49 = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot \cancel{48} \cdot \cancel{47} \cdot \cancel{46} \cdot \cancel{45} \cdot \cancel{44} \cdot \cancel{43} \cdot \cancel{42} \cdot \cancel{41} \cdot \cancel{40} \cdot \cancel{39} \cdot \cancel{38} \cdot \cancel{37} \cdot \cancel{36} \cdot \cancel{35} \cdot \cancel{34} \cdot \cancel{33} \cdot \cancel{32} \cdot \cancel{31} \cdot \cancel{30} \cdot \cancel{29} \cdot \cancel{28} \cdot \cancel{27} \cdot \cancel{26} \cdot \cancel{25} \cdot \cancel{24} \cdot \cancel{23} \cdot \cancel{22} \cdot \cancel{21} \cdot \cancel{20} \cdot \cancel{19} \cdot \cancel{18} \cdot \cancel{17} \cdot \cancel{16} \cdot \cancel{15} \cdot \cancel{14} \cdot \cancel{13} \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{48} \cdot \cancel{47} \cdot \cancel{46} \cdot \cancel{45} \cdot \cancel{44} \cdot \cancel{43} \cdot \cancel{42} \cdot \cancel{41} \cdot \cancel{40} \cdot \cancel{39} \cdot \cancel{38} \cdot \cancel{37} \cdot \cancel{36} \cdot \cancel{35} \cdot \cancel{34} \cdot \cancel{33} \cdot \cancel{32} \cdot \cancel{31} \cdot \cancel{30} \cdot \cancel{29} \cdot \cancel{28} \cdot \cancel{27} \cdot \cancel{26} \cdot \cancel{25} \cdot \cancel{24} \cdot \cancel{23} \cdot \cancel{22} \cdot \cancel{21} \cdot \cancel{20} \cdot \cancel{19} \cdot \cancel{18} \cdot \cancel{17} \cdot \cancel{16} \cdot \cancel{15} \cdot \cancel{14} \cdot \cancel{13} \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$${}_{20}P_3 = 20 \cdot 19 \cdot 18 = \frac{20!}{?} = \frac{20!}{17!}$$

Formula for ${}_nP_r$: ${}_nP_r = \frac{n!}{(n-r)!}$

5

On your calculator

$${}_7P_3$$

- TI-83

7 - MATH - PRB - nPr - 3 - ENTER

6

Combinations - Order Does Not Matter

- The Classical Studies Department has 7 faculty members. Three must attend the graduation ceremonies. How many different groups of 3 can be chosen?
- If order mattered, the answer would be $7 \cdot 6 \cdot 5 = 210$
- Let's look at one set of three professors: A, B, C:
A B C A C B B C A B A C C A B C B A
- Why are there 6 listings for the same set of 3 profs?

There are $3! = 6$ possible arrangements of three objects.

7

nPr

Recall the convention: $0! = 1$

There are $3 \cdot 2 \cdot 1 = 3!$

arrangements of 3 objects.

Using the nPr notation, from a set of 3 objects we are choosing 3.

$${}_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3!$$

8

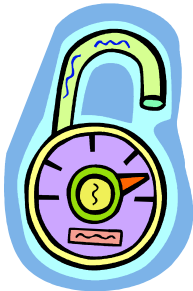
Combinations: ${}^7C_3$

- In our list of 210 sets of 3 professors, with order mattering, each set of three profs is counted $3! = 6$ times. The number of distinct **combinations** of 3 professors is

$$\begin{aligned} {}^7C_3 &= \frac{{}^7P_3}{6} = \frac{{}^7P_3}{3!} \\ &= \frac{7!}{(7-3)!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{210}{6} = 35 \end{aligned}$$

${}^7C_3$ is the number **combinations** of 3 objects chosen from a set of 7. “Of seven, take three”

9



Warning

The “combination,” or sequence of three numbers, on your combination lock is **not** a combination ! Order matters!

10

nCr

- Factorial formula is: $nCr = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$

- Practice:

$$8C_2$$

$$8C_6$$

$$10C_4$$

11

nCr (of n , pick r)

- Factorial formula is: $nCr = \frac{n!}{r!(n-r)!}$

- Practice:

$$8C_2 = \frac{8!}{2!6!} = \frac{8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{2 \cdot 1 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

$$8C_6 = \frac{8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 28$$

$$10C_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

12

What about xP_3 and xC_3 ?

$${}_xP_3 = x(x-1)(x-2)$$

$${}_xC_3 = \frac{x(x-1)(x-2)}{3!}$$

13

Application to poker: Full House



In how many different ways can one have a full house with sixes over nines?

ways to have 3 sixes	=	ways to have 2 nines	=	$4C_3$	=	$4C_2$	=	$4 \cdot 6 = 24$
----------------------------	---	----------------------------	---	--------	---	--------	---	------------------

14

How many ways to have a Full House?

1. How many ways to have a full house with “sixes over”?

There are 24 ways to have 6's over 9's.

Could also have 6's over 2's, 6's over 3's, etc

12 possibilities for the pair

$$24 * 12 = 288 \text{ ways to have a full house with 6's over.}$$

2. How many ways to have a full house?

Could have 7's over, Queens over, etc..

13 possibilities for the triplet

$$13 * 24 * 12 = 3744 \text{ ways to have a full house}$$

15

Stock Market Example

Suppose 12 stocks have been traded, and

7 increased, 3 decreased, 2 stayed the same.

In how many ways could this happen?

Think: 3 separate selections: “three-box problem”

Of the 12,
7 increased

Of the
remaining 5,
3 decreased

Of the
remaining
2, 2 stayed
the same

$$12C7$$

$$5C3$$

$$2C2 = 1$$

$$\frac{12!}{7!5!} \frac{5!}{3!2!} 1 = \frac{12!}{7!3!2!} = 7920$$

16

How many 3-digit numbers can be made using only 2 different digits?

Number must have 2 of one digit, one of another

ways to select
repeated digit

10

ways to select
single digit

9

= 90

Once selection is made, how many arrangements are possible?

Suppose we'd selected: 4 4 5

How many places to put the "5"?

How many ways to choose 1 of 3 possible locations? ${}_3C_1 = 3$

90 selections, 3 arrangements for each: $90 * 3 = 270$

17

There are many ways to do these problems!

- Some are more complicated than others.
- Sometimes there are shortcuts.
- Sometimes there are not.

18

Rearrangements with repeated letters; example BOSS

There are $4!/2! = 12$ distinct rearrangements of the letters BOSS

4! arrangements if S and S are distinct

B O S S

BOSS	OBSS	SSBO	~~SSBO~~
~~BOSS~~	~~OBSS~~	SSOB	~~SSOB~~
BSOS	OSBS	SBOS	~~SBOS~~
~~BSOS~~	~~OSBS~~	SBSO	~~SBSO~~
BSSO	OSSB	SOBS	~~SOBS~~
~~BSSO~~	~~OSSB~~	SOSB	~~SOSB~~

If not, cross out repeats

There are $2! = 2$ rearrangements of S and S

Every distinct arrangement had been counted 2 times. Divide by 2.

19

STRESS

How many distinct arrangements if **S**, **ſ**, and **Ɲ** are regarded as distinct letters? $6! = 720$

What if **S**, **ſ**, and **Ɲ** are **not** regarded as distinct letters? If all the fonts were changed to Arial, how many of the 720 would look like **STRESS**?

S TRE **ſ** S **S** TRES **Ɲ** **S** TRE **S** **S**

How many ways can one rearrange **S**, **ſ**, and **Ɲ**? $3! = 6$

The number of distinct rearrangements of STRESS is

$$\frac{720}{6} = \frac{6!}{3!} = 120$$

20

Rearrangements of sets containing repeats

- STRESS

$$\frac{6!}{3!} = 120$$

- STRESSED

$$\frac{8!}{3!2!} = 60$$

STRESSED

If we could distinguish the E's, $\frac{8!}{3!}$

- SUPERSTRESSED

4 S's, 3 E's, 2 R's

$$\frac{13!}{4!3!2!} = 21621600$$

21