Chapter 1
Functions and Limits

Contents

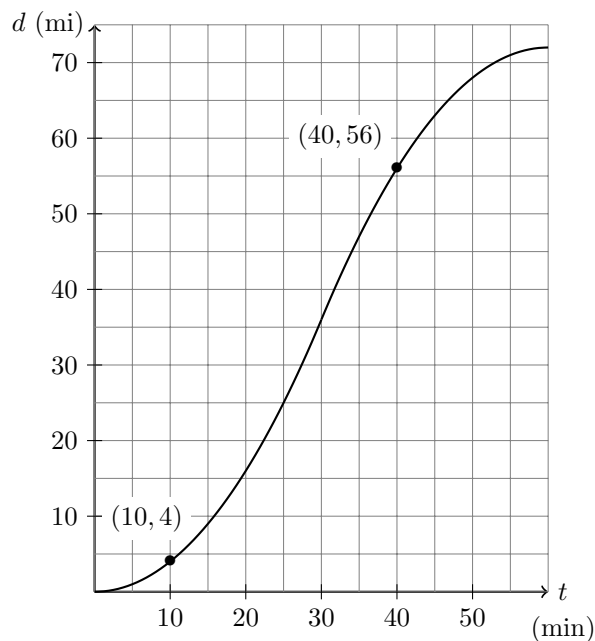## Motivation to Chapter 1

The first big topic of calculus is slope. This is an extremely important topic not just for math but across all of the sciences. Let's motivate it with an example.

Example: You are driving from Lansing to Detroit. To the right is a graph representing your distance from Lansing. What is your average velocity on the interval $[10, 40]$ on your trip?

Example: At what time is your speed the greatest?

Slope helps us determine how fast things are changing, in this case how fast the distance is changing (aka velocity). More generally the (average) rate of change is something worth studying, not just for velocity sake but so much more! Other examples include:

- Rate of increase/decrease of stock values.

- Infection rate of a disease.

- Rate of a chemical reaction.

- Rate of population growth.

- Rate of change of velocity (acceleration).

- Rate of change for monthly sales at your business.

- Rate at which a temperature changes.

4   Tangent and Velocity Problems

4.1   VIDEO - Secant Lines, Average Rate of Change, and Applications

Objective(s):

- Be able to visualize secant lines.

- Compute the slope of secant lines.

- Apply secant lines to real world problems.

As we saw in the motivation section that how a quantity changes is important. Let's start to define this more precisely

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Definition(s) 4.1. The _____ of $y = f(x)$ through $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ is given by

Alternatively, this is referred to as the _____ on the interval _____ .

And let's visualize it!

Secant Line
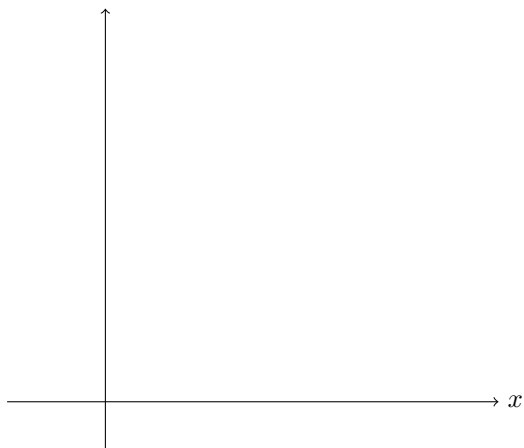
- Need a _____ .

- Need _____ specified point(s).

Now I think we have a pretty good idea of how these things work but it is really nice to be able to play with them so we can use the applet `https://www.desmos.com/calculator/kktiplmtbk` to really help us visualize secant and average rates of change.

Example 4.2. Let $f(x) = \sqrt{x+1}$. Find the slope of the secant line joining the points $(3, f(3))$ and $(8, f(8))$.

Alternate Phrasing: *Calculate the average rate of change of $f(x) = \sqrt{x+1}$ on the interval $[3, 8]$.*

Year	Board Games Sold
2010	512
2011	601
2012	943
2013	1120
2014	1342
2015	1854
2016	2612

Example 4.3. In the year 2010 Ryan started a business *Ryan's Friendly Board Game Emporium.* To the left is a table of the number of board games I have sold each year. Find the average rate of change during 2010 to 2016 and interpret the result.

Remark 4.4. The average rate of change of $f(x)$ has units of:

4.2 VIDEO - Tangent Lines, Instantaneous Rate of Change, and Applications

Objective(s):

- Be able to visualize tangent lines.
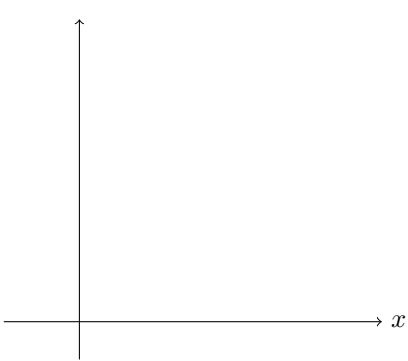
- Approximate the slope of tangent lines lines.

When I was much younger I was pulled over for speeding. The police officer had used a radar gun and determined that at a particular instant I was going 56 miles per hour. This brings up an idea related to average rate of change but at one specific time. A phrase we will become familiar with is instantaneous rate of change, that is how a function is changing at a particular instant of time (just like how I was going 56 miles per hour).

In our last video we use secant lines to help visualize average rates of change. Now we will use something called

_____ to help visualize instantaneous rate of change. But it turns out that tangent lines are pretty hard to give a definition to (at least at this point). So let's sketch and play and go over some properties of tangent lines

Tangent Line

- Need a _____.

- Need _____ specified point(s).

- Just barely _____ the graph of the function.

- It's a line close to the secant line through two

   _____.

For some visualization let's go back to our applet `https://www.desmos.com/calculator/kktiplmtbk`.

Example 4.5. Use the Desmos site to approximate the slope of the tangent line of $f(x) = x^2 - 3$ at $x = 2$.

Remark 4.6. Just as the slope of the secant line represents the average rate of change, the slope of the tangent line represents the _____.

Of course there is only so much we can do in this class without a calculator.

Example 4.7. If a rock is thrown upward on the planet Mars with an initial velocity of 10 meters per second, its height (in meters) t seconds later is well approximated by $h(t) = 10t - 2t^2$.

(a) Find the average velocity over the time interval $[1, 2]$.

(b) Find the average velocity over the time interval $[0, 1]$.

(c) Using part (a) and (b) approximate the instantaneous velocity at $t = 1$.

Remark 4.8. The instantaneous rate of change of $f(x)$ have units of:

5   The Limit of a Function

5.1   VIDEO - A Graphical Approach

Objective(s):

- Have an intuitive idea of the definition of a limit.

- Find the limits (two-sided, left, and right) of the piecewise defined function given algebraically or graphically.

Definition(s) 5.1. Suppose _____ is defined when x is near the

_____. Then we write:

if we can force the values of $f(x)$ _____ as we like to L by

requiring x to be sufficiently _____ (but not equal) to a.

Definition(s) 5.2 (Left-hand limit). We write

if we can force the values of $f(x)$ _____ as we like to L by

requiring x to be sufficiently _____ to a and x is less than a.

Definition(s) 5.3 (Right-hand limit). We write

if we can force the values of $f(x)$ _____ as we like to L by

requiring x to be sufficiently _____ to a and x is greater than a.

Let's take a closer look using: `https://www.desmos.com/calculator/zpdfmssol9`

Theorem 5.4.

if and only if

Example 5.5. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim\limits_{x \to 2^-} f(x)$

(b) $\lim\limits_{x \to 2^+} f(x)$

(c) $\lim\limits_{x \to 2} f(x)$

(d) $\lim\limits_{x \to 4} f(x)$

(e) $f(2)$

(f) $f(4)$
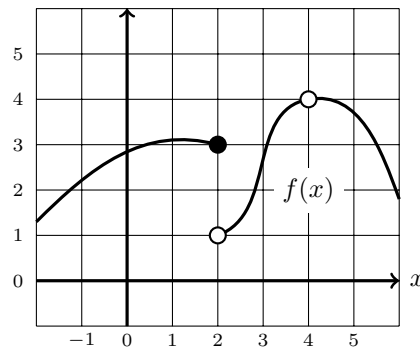
Example 5.6. Evaluate the following limits for the function $f(x) = \begin{cases} x^2 + 4 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 3 - x & \text{if } x > 1 \end{cases}$

(a) $\lim\limits_{x \to 1^-} f(x)$

(b) $\lim\limits_{x \to 1^+} f(x)$

(c) $\lim\limits_{x \to 1} f(x)$

(d) $\lim\limits_{x \to 2} f(x)$

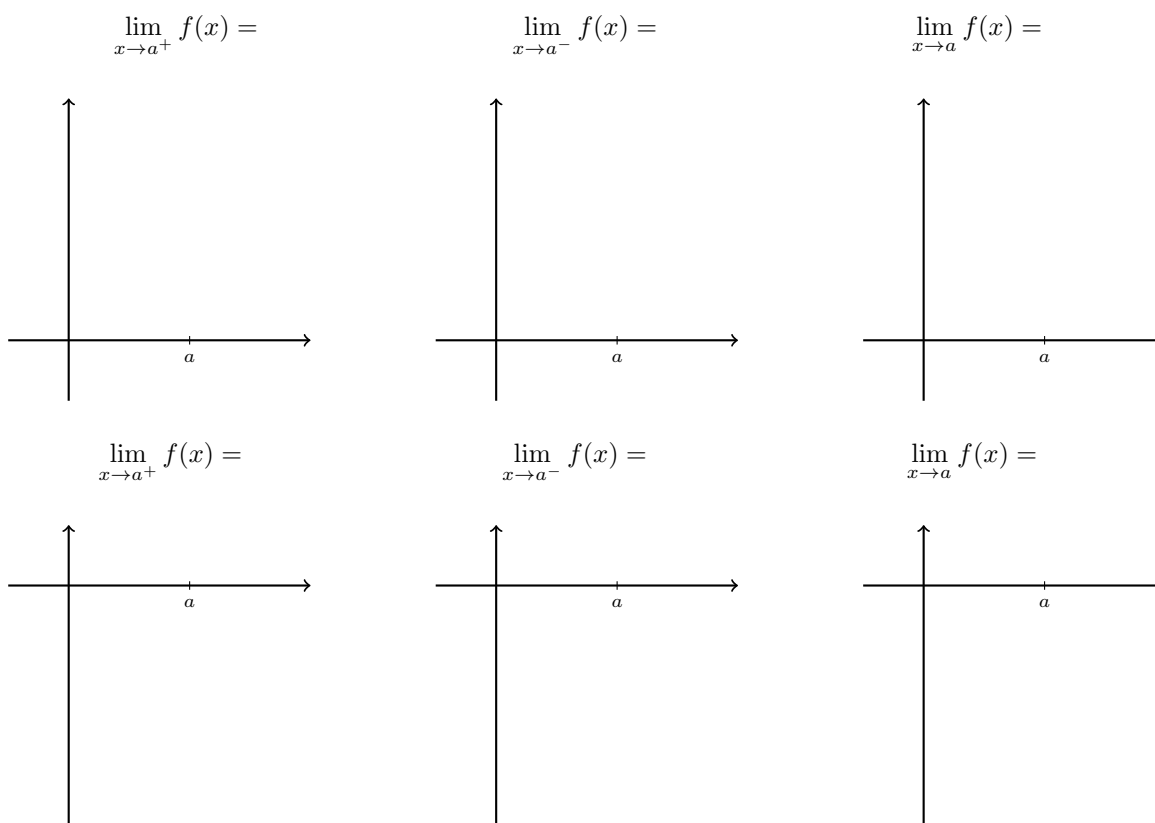## 5.2　VIDEO - Evaluating Infinite Limits Numerically

Objective(s):

- Calculate infinite limits and detect vertical asymptotes.

By now in your mathematical career you have probably encountered vertical asymptotes such as in $y = \dfrac{1}{x}$ or in $y = \tan \theta$.
Let's do a little review of $\tan \theta$ to help motivate vertical asymptotes.

Example 5.7. Explore $y = \tan \theta$

Definition(s) 5.8. The line $x = a$ is called a _____ of the curve $y = f(x)$ if at least

one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \qquad\qquad \lim_{x \to a^-} f(x) = \qquad\qquad \lim_{x \to a} f(x) =$$

$$\lim_{x \to a^+} f(x) = \qquad\qquad \lim_{x \to a^-} f(x) = \qquad\qquad \lim_{x \to a} f(x) =$$

That is f can be made arbitrarily _____ (or _____ with _____) by taking x sufficiently close to a, but not

equal to a.

Remark 5.9. For functions of the form $\dfrac{g(x)}{h(x)}$ where $g(a) \neq 0$ and $h(a) = 0$ often there is a vertical asymptote. To verify compute the left and right hand limits as $x \to a$.

Example 5.10. Find the vertical asymptotes of the function $y = \dfrac{x^2 + 2x + 1}{x + x^2}$

Example 5.11. Evaluate $\displaystyle\lim_{x \to 1^+} \dfrac{5}{1 - x}$

Example 5.12. Evaluate the limit $\displaystyle\lim_{x \to 0^-} \dfrac{x^2 + 2x + 1}{x + x^2}$

6   Calculating Limits Using the Limit Laws

6.1   VIDEO - Case 1 (0/N and N/N): Simple Limit Laws

Objective(s):

- Utilize limit laws to calculate other limits.

- Apply the direct substitution property to more complicated limits.

Theorem 6.1 (Limit Laws). Suppose that c is a constant and the limits

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then:

1. $\lim_{x \to a} [f(x) + g(x)] =$

2. $\lim_{x \to a} [cf(x)] =$

3. $\lim_{x \to a} [f(x)g(x)] =$

4. $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)} \right] =$ provided $\lim_{x \to a} g(x) \neq 0$

5. $\lim_{x \to a} c =$

6. $\lim_{x \to a} x =$

7. $\lim_{x \to a} x^n =$ when appropriate.

8. $\lim_{x \to a} [f(x)^n] =$ when appropriate.

Example 6.2. Evaluate the limit $\lim_{x \to 0} \dfrac{2x^2 - x}{3 + x}$

Theorem 6.3 (Direct Substitution Property)**.** If f is a polynomial, a rational function, or a root function and a is in the domain of f then

Remark 6.4. Because of this wonderful theorem the first step to evaluate a limit is to plug in an a everywhere there is an x to determine what case you are in!

Example 6.5. Evaluate $\lim\limits_{x \to -3} \dfrac{\sqrt{1-x}}{x^2 - 4x}$

Example 6.6. If $\lim\limits_{x \to 2} f(x) = 3$ and $\lim\limits_{x \to 2} g(x) = -2$ then find $\lim\limits_{x \to 2} [5f(x) - xg(x)]$.

6.2   VIDEO - Case 2 (N/0): Vertical Asymptotes as in Section 5

Example 6.7. Evaluate $\lim\limits_{x \to 1^+} \dfrac{-4}{1-x}$

6.3 VIDEO - Case 3 (0/0): Applying Algebra to Limits

Objective(s):

- Recall old algebraic techniques of factoring, multiplying by a conjugate, etc. Use these to calculate a variety of limit problems.

Theorem 6.8 (Cancellation). If $f(x) = g(x)$ when $x \neq a$, then

provided the limit exists.

Example 6.9. Evaluate the limit, if it exists $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 3}{x^2 - 1}$

Example 6.10. Evaluate the limit, if it exists $\lim\limits_{x \to 2} \dfrac{\dfrac{1}{x-1} - 1}{x - 2}$

Example 6.11. Evaluate the limit, if it exists $\displaystyle\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$

6.4   VIDEO - Limits and Inequalities

Objective(s):

- Comprehend and utilize the Squeeze Theorem to evaluate more complicated limits.

We have seen that by using some algebraic techniques; such as factoring, common denominators, or multiplying by a conjugate that we can solve many $\dfrac{0}{0}$ limit problems. Although we have come a long way we can still not solve

Example 6.12. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

and so we need another technique!

Theorem 6.13. If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

Theorem 6.14 (Squeeze Theorem). If $f(x) \leq g(x) \leq h(x)$ when

x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

Example 6.15. Suppose $2x + 1 \leq g(x) \leq x^2 - 2x + 5$ for all x. Find $\lim_{x \to 2} g(x)$

Example 6.16. Use the Squeeze Theorem to find $\lim_{x \to 0} \left[x^2 \cos\left(\frac{2}{x}\right) \right]$

8 Continuity

8.1 VIDEO - What When and Why of Continuity

Objective(s):

- Detect when a function is continuous and when it is discontinuous.

- Use continuity to quickly evaluate limits.

Back in the day we used to say "*A function is continuous if you can draw it without lifting your pencil.*" and while this is true and a great intuitive definition we can now be a bit more precise (thanks to our knowledge of limits).

Example 8.1. Consider the graph of $f(x)$ to the right on the domain $[-2, 5]$. Determine where the function is continuous.
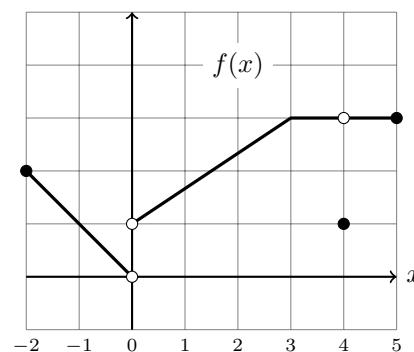
Definition(s) 8.2. A function f is _____ if and only if

Note additional subtleties : This requires

1. _____ to be defined

2. _____ to exist.

Definition(s) 8.3. A function f is continuous on an interval if it is continuous at each point in that interval.

Theorem 8.4. Essentially all the functions we love (polynomial, rational, root, triginometric) are continuous _____ in their _____.

Example 8.5. Evaluate $\lim\limits_{x \to 0} \dfrac{x^2 + 3x - 4}{x^2 - 1}$

Example 8.6. Evaluate $\lim\limits_{t \to \pi} 5\cos(t)$

A next natural question one could ask is "*What about combinations of our beloved functions?*" so things like:

Example 8.7. Evaluate $\displaystyle\lim_{x\to 1} \frac{\sin(x^2+1)}{\sqrt{x}(5-x)}$

and the answer is.... _____ the time we _____. More formally,

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Theorem 8.8. If f and g are continuous at a and c is constant, then the following are also continuous at a.

1.   2.   3.

4.   5.

Theorem 8.9. If g is continuous at ___ and f is continuous at _____ , then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is _____.

Example 8.10. Determine where $g(x) = \dfrac{\sqrt{1-x^2}}{x}$ is continuous.

8.2   VIDEO - Oops Not Continuous

Objective(s):

- Explain and classify the different types of discontinuities.

Remark 8.11. So as we saw above lots of functions are continuous (in their domains). The things to watch out for are

(a) Dividing _____ .

(b) Square roots of _____ .

(c) _____ defined functions.

(d) Graphs of functions.

Example 8.12. Consider the function $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ 2\cos(\pi x) & \text{if } 0 \le x \le 2 \,. \\ 6 - x^2 & \text{if } x > 2 \end{cases}$

Determine value(s) of x at which $f(x)$ is discontinuous.

Definition(s) 8.13. If f is defined near a (except perhaps at a), we say that f is

_____ (or f has a _____ at a) if f is not continuous at a.

Definition(s) 8.14.

(a) A _____ is a discontinuity that can be _____ by redefining f

at just a _____ .

(b) An _____ another name for a vertical asymptote.

(c) All others in this class are called _____ discontinuities. In these cases you will notice the function

_____ from one value _____ .

Pictures
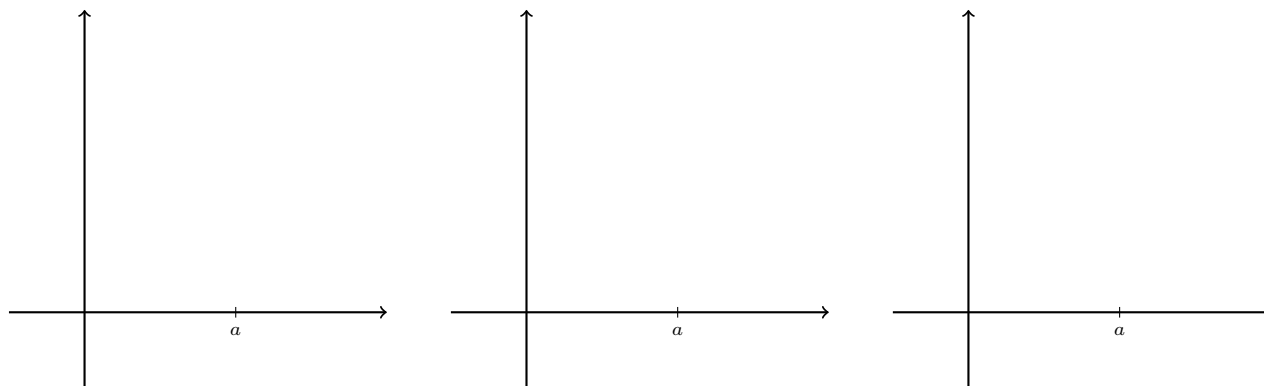
Remark 8.15. Here are how these discontinuities are most likely to appear:

(a) Removable discontinuities typically occur in the _____ (back in 1.6).

(b) Infinite discontinuities typically occur in the _____ (back in 1.6/1.5).

(c) Jump discontinuities are typically occur in _____ defined functions.

There are some exceptions but I find this holds true 90%+ of the time.

Remark 8.16. The act of *removing* a removable discontinuity usually comes down to _____ the function algebraically.

Example 8.17. Determine and classify the discontinuities of $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 3x - 10}$.

Example 8.18. $f(x) = \dfrac{3x^2 - 3x}{x^2 - 1}$ has a removable discontinuity at $x = 1$. Find a function $g(x)$ that agrees with $f(x)$ for $x \neq 1$ and is continuous at 1.

8.3   VIDEO - Continuity and the IVT

Objective(s):

- Understand the statement of the Intermediate Value Theorem

- Apply the Intermediate Value Theorem to mathematically prove two functions intersect on a set interval.

Theorem 8.19 (Intermediate Value Theorem (IVT))**.** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.

Picture:
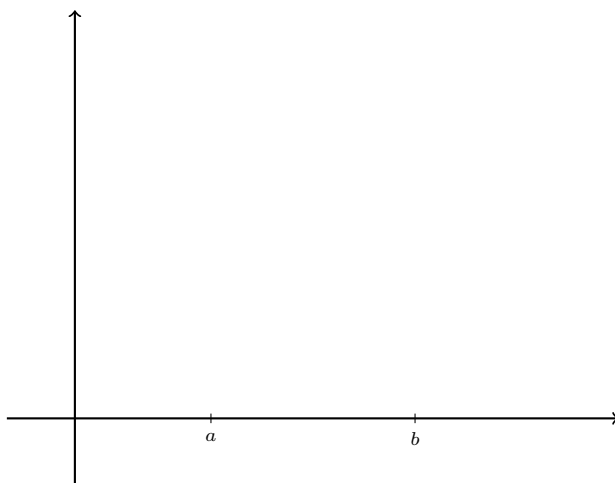
Remark 8.20. The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

Example 8.21. Use the Intermediate Value Theorem to show that there is a root of the function:

$f(x) = x^4 + x - 3$ on the interval $(1, 2)$.

Remark 8.22. Notice that it is very difficult to solve $x^4 + x - 3 = 0$!

Example 8.23. Suppose $f(x)$ is a continuous function with values given by the table below.

x	0	1	2	3	4	5
$f(x)$	10.1	3.4	2.9	-1.5	0	0.8

On which interval **must** there be a c for which $f(c) = 4$?

   A. $(0, 1)$

   B. $(1, 2)$

   C. $(2, 3)$

   D. $(3, 4)$

   E. $(4, 5)$

Example 8.24. Prove that the equation $\cos x = x^3$ has at least one solution. What interval is it in?