Chapters 4 & 5
Integrals & Applications

Contents

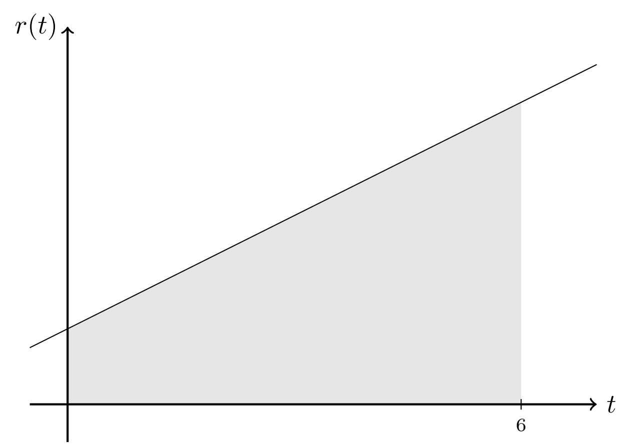
Motivation to Chapters 4 & 5

In Chapters 4 and 5 we will bring up the second main topic: area 'under' (the graph of) a function. We will see that surprisingly this is related to derivatives. We will learn how to find the area 'under' more complicated functions and the area trapped between two functions. But why do we care?

Well as usual it's because there are real world applications.

Remark 0.1. If $f(x)$ is a ________________________ then the area under $f(x)$ on the interval $[a, b]$ is the _____________________ of from $x = a$ to $x = b$.

Example 0.2. A pump is delivering water into a tank at a rate of $r(t) = 1 + \frac{t}{2}$ liters/minute, where t is time in minutes since the pump was turned on. How much water has been pumped in the first 6 minutes?



And so anytime you have a rate of change and you are interested in total change you will want area under functions.

- You have velocity and want displacement
- You have a rate of change of an infection and want the total number of people infected
- You have marginal profit and want total profit.

1 Areas and Distances

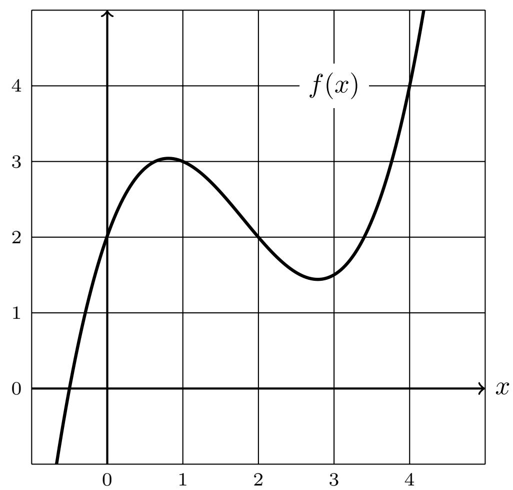
1.1 VIDEO - Areas Under Functions

Objective(s):

- Estimate the area under a curve using rectangles with heights given by left endpoints or right endpoints.
- Solve for over/under estimates for the area under a curve using rectangles.

Consider the following problem which helps to emphasize some issues

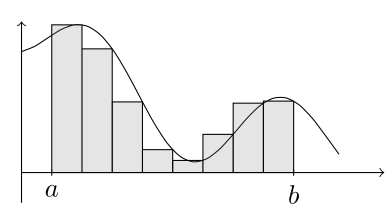
Example 1.1. Find area between $f(x)$ and the $x-$axis on the interval $[1, 4]$.



Example 1.2. Approximate the area between $f(x)$ and the $x-$axis on the interval $[1, 4]$ using 3 rectangles of equal width.

The goal of this section is to understand how to approximate areas with rectangles. Suppose that we want to approximate the area between the graph of a continuous function $f(x)$ and the x-axis between $x = a$ and $x = b$ (suppose for now that f is positive). Let's sub-divide the interval $[a, b]$ into n sub-intervals $[x_0, x_1]$ through $[x_{n-1}, x_n]$ of equal width Δx. If we pick a point x_i^* in each interval $[x_i, x_{i+1}]$, then we can estimate the area under the graph by the sum of the areas of the rectangles with width Δx and height $f(x_i^*)$:

$$\text{Area} \approx f(x_0^*)\Delta x + f(x_1^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x$$



But the question remains of what x_i^* are the should we pick?

Definition(s) 1.3.

- An ______________________ is when the x_i^* are all chosen to be the global maximum on $[x_i, x_{i+1}]$ for each i.
- A ______________________ is when the x_i^* are all chosen to be the global minimum on $[x_i, x_{i+1}]$ for each i.
- A ______________________________ is when x_i^* is the left endpoint of $[x_i, x_{i+1}]$ for each i.
- A _________________________________ is when x_i^* is the right endpoint of $[x_i, x_{i+1}]$ for each i.

Example 1.4. Consider the continuous function $g(x)$ whose values are given in the table below

x	0	0.5	1.0	1.5	2.0
$g(x)$	1	6	4	2	5

Assume all absolute maximums and minimums of $g(x)$ are contained in the above points.

(a) Using 4 rectangles find the upper sum of $g(x)$ on the interval $[0, 2]$.

(b) Using rectangles of width 0.5 find the lower sum of $g(x)$ on the interval $[0, 2]$.

(c) Using $\triangle x = 0.5$ find the left-hand sum of $g(x)$ on the interval $[0, 2]$.

(d) Using $n = 4$ find the right-hand sum of $g(x)$ on the interval $[0, 2]$.

1.2 VIDEO - Applications of Area

Objective(s):

- Argue that area under a curve of a velocity function will yield displacement.
- Solve various application problems.

You may have learned in physics class that if you have a constant rate (think velocity) then

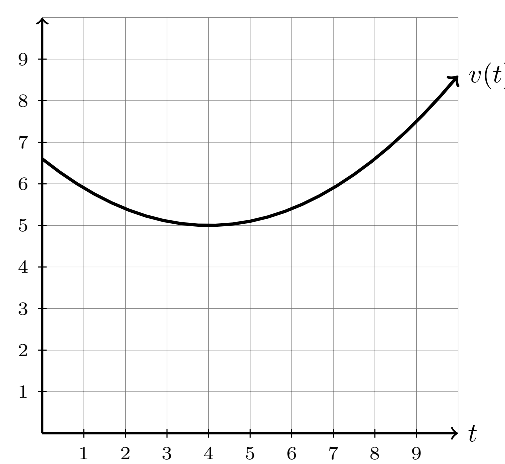
$$\text{distance} =$$

Example 1.5. Suppose a runner travels at 20 feet per second.

How far does she travel between $t = 1$ seconds and $t = 9$ seconds?

And so we see that actually that the distance (or more accurately displacement) is the area 'under' the velocity graph. And in this way we can extend this idea to more realistic situations.

Example 1.6. Suppose a runner's velocity (in ft/s) is given by the curve to the right. Sketch the area that represents how far she traveled between $t = 1$ seconds and $t = 9$ seconds.



Example 1.7. Approximate the distance the runner traveled between $t = 1$ and $t = 9$ seconds using a left-hand sum and 4 rectangles. Sketch this area on the graph above

Example 1.8. Suppose the velocity of a runner is given by $v(t) = \sqrt{t}$ meters per second. Approximate the distance the runner traveled from $t = 0$ to $t = 4$ seconds.

(a) Using time intervals of 1 second and a left-hand sum

(b) Using time intervals of 1 second and a right-hand sum

2 The Definite Integral

2.1 VIDEO - Where Are We Going With This?

Objective(s):

- Learn how to get better approximations for area under a curve.
- Setup why we want to be able to add together 100, 1,000, or 1,000,000 numbers

For a bit now we have been approximating the area under a function using areas of rectangles. Now we face the issue of accuracy. Mainly, if you want to be as accurate as possible how many rectangles should you use? The answer.... with __________ rectangles comes __________ accuracy.

Check out `https://www.desmos.com/calculator/cxsfmpvf69` to see this in action.

Okay so we want more rectangles... so what? Well let's try a problem to see what challenges we will face.

Example 2.1. Approximate the area under $f(x) = x$ on the interval $[1, 2]$ using right-hand sums and 100 rectangles.

2.2 VIDEO - Sigma Notation

Objective(s):

- Express sums using sigma notation.
- Memorize a few common finite sums.
- Understand basic properties of finite sums and use them to compute more complicated finite sums.

Definition(s) 2.2. If $a_m, a_{m+1}, \ldots, a_{n-1}, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i =$$

Definition(s) 2.3. This way of short-handing sums of many numbers is called ________________________ (uses the Greek letter Σ "Sigma"). The letter i above is called the ________________________________ and it takes on consecutive integer values starting with m and ending with n.

Theorem 2.4. If c is any constant then:

(a) $\displaystyle\sum_{i=m}^{n} ca_i =$

(b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) =$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) =$

Theorem 2.5. Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 =$

(b) $\displaystyle\sum_{i=1}^{n} i =$

(c) $\displaystyle\sum_{i=1}^{n} i^2 =$

Example 2.6. Write the sum: $\sqrt{3}+\sqrt{4}+\cdots+\sqrt{25}$ in sigma notation

Example 2.7. Evaluate the following sums

(a) $\displaystyle\sum_{i=1}^{4}(2-3i)$

(b) $\displaystyle\sum_{i=1}^{400}(2-3i)$

2.3 VIDEO - The Definite Integral

Objective(s):

- Use the limit of finite sums to calculate the definite integral of a function.
- Identify how the definite integral relates with area under the curve.

If more rectangles are good then why not have 1000, 100000, or ideally even infinitely many rectangles.

Then the area would be so super accurate that there wouldn't be any difference at all!

Example 2.8. Using 8 rectangles and a right-hand sum approximate the area under the curve $f(x) = \sqrt{x}$ on $[1, 4]$.

Do not simplify anything.

Now to make the jump to infinitely many rectangles let's start off by calculating the area under n rectangles.

Example 2.9. Using n rectangles and a right-hand sum approximate the area under the curve $f(x) = \sqrt{x}$ on $[1, 4]$.

Do not simplify anything.

Remark 2.10.

(a) The width of a rectangle is given by $\Delta x =$

(b) The right-hand endpoint of the i^{th} rectangle is given by $x_i =$

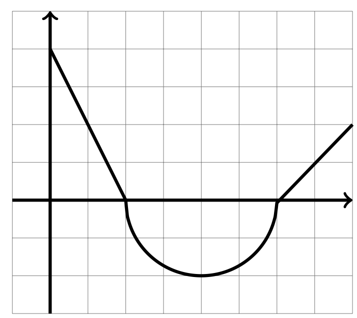
Definition(s) 2.11 (Definite Integral)**.** If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx =$$

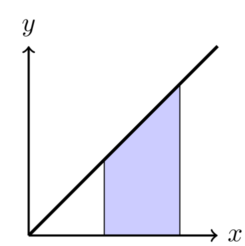
Remark 2.12. The definite integral, $\int_a^b f(x)\, dx$ gives the ________________ between the curve f and the $x-$axis.

That is, if the function is __________ the $x-$axis the area is counted _________________.

Example 2.13. Evaluate the definite integral $\int_1^7 f(x)\, dx$. Where $f(x)$ is given by the function to the right.



Example 2.14. Use the definition of the definite integral to evaluate $\int_1^2 x\, dx$.



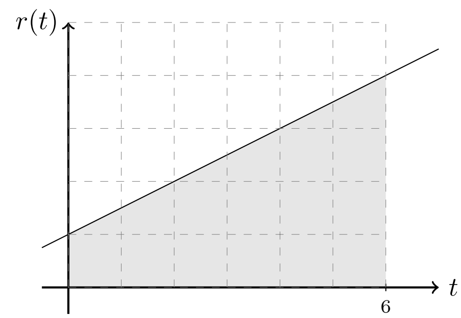
3 The Fundamental Theorem of Calculus

3.1 VIDEO - Part 1 of the FToC

Objective(s):

- State the first part of the FToC.
- Go over an idea of the proof to see why it is probably true.
- Apply the FToC, properties of definite integrals, and the chain rule to take some derivatives.

Example 3.1. A pump is delivering water into a tank at a rate of $r(t) = 1 + \frac{t}{2}$ liters/minute, where t is time in minutes since the pump was turned on. What does $w(x) = \int_0^x r(t)\, dt$ represent? Evaluate $w(1)$ and $w(3)$.



Theorem 3.2 (FTC, Part 1)**.** If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \qquad\qquad\qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) =$ ________

Remark 3.3. Here is an idea of the proof:

Example 3.4. Find the derivative of $g(x) = \int_1^x t^2 \, dt$

Example 3.5. Find the derivative of $f(x) = \int_x^2 \sqrt{t^2+3} \, dt$

Example 3.6. Find the derivative of $H(x) = \int_{-1}^{x^3} \sin(2t^2) \, dt$

3.2 VIDEO - Part 2 of the FToC

Objective(s):

- State the second part of the FToC.
- Go over an idea of the proof to see why it is probably true.
- Use the antiderivative to calculate definite integrals.

Theorem 3.7 (FTC, Part 2)**.** If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx =$$

where F is any antiderivative of f, that is, a function such that ____________.

Remark 3.8. Here is an idea of the proof:

Remark 3.9. The two parts of the FTC together state that differentiation and integration are ____________ processes.

Remark 3.10. From our perspective **FTC, Part** ___ is the most important because it allows us to calculate definite integrals without having to take limits of Riemann sums!

Notation 3.11. $F(b) - F(a)$ will also be denoted $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

Example 3.12. Evaluate the following integrals

(a) $\displaystyle\int_1^9 \sqrt{x}\, dx$

(b) $\displaystyle\int_{-1}^5 (u+1)(u-2)\, du$

(c) $\displaystyle\int_1^2 \frac{x^4+1}{x^2}\, dx$

4 Indefinite Integrals and Net Change

4.1 VIDEO - Indefinite Integrals

Objective(s):

- Define Indefinite integrals and introduce some of their properties.
- Practice computing indefinite integrals.

Definition(s) 4.1. An ______________________________ of f, written $\int f(x)\,dx$, is an ____________________ of f. In other words,

$$F(x) = \int f(x)\ dx \qquad \text{means}$$

Theorem 4.2 (Linearity Properties of Indefinite Integrals). For functions $f(x)$ and $g(x)$, and any constant $k \in \mathbb{R}$,

$$\int [f(x) + g(x)]\,dx =$$

and

$$\int kf(x)\ dx =$$

Example 4.3. Find the most general function $F(x)$ with the property that:

(a) $F'(x) = \dfrac{1}{\sqrt{x}} + \cos(x)$

(b) $F'(x) = \sec(x)\,(\tan(x) + \sec(x))$

4.2 VIDEO - Net Change Theorem

Objective(s):

- Use the Net Change Theorem to continue calculating definite integrals.
- Given a velocity function and a set time interval find the total distance traveled.

Theorem 4.4 (**Net Change Theorem**). The net change of a quantity $F(x)$ from $x = a$ to $x = b$ is the integral of its derivative from a to b. That is, if $F'(x) = f(x)$, then

$$= \int_a^b F'(x)\ dx = \int_a^b f(x)\ dx$$

In other words,

$$= F(a) + \int_a^b F'(x)\ dx = F(a) + \int_a^b f(x)\ dx$$

Remember that definite integrals are sums and the derivative is the instantaneous rate of change. So the Net Change Theorem says if you sum up the instantaneous rates of change you get total change. And when you put it like that it sounds almost obvious.

Remark 4.5. If $s(t)$ represents the position of a moving object, and $v(t)$ the velocity, then the **Net Change Theorem** says we can compute the net displacement of the object from time $t = a$ to time $t = b$ by integrating the velocity:

$$\int_a^b s'(t)\ dt = \int_a^b v(t)\ dt =$$

Example 4.6. An object moves along a line with velocity $v(t) = 8t - t^2$ inches per second. At time $t = 0$, the object is 5 inches to the right of the origin. What is the position of the object at time $t = 6$ seconds?

5 The Substitution Rule

5.1 VIDEO - Indefinitely

Objective(s):

- Develop a substitution rule to find antiderivatives of more complicated functions.
- Apply it to a couple of problems.

Theorem 5.1. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

Remark 5.2. The Substitution Rule says essentially that is is permissible to operate with dx and du after the integral signs as if they were differentials.

Example 5.3. Evaluate the following indefinite integrals:

(a) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

(b) $\displaystyle\int \frac{dt}{(1-3t)^5}$

5.2 VIDEO - Definitely

Objective(s):

- Upgrade our use of the substitution rule to include definite integrals.
- Apply the substitution rule to more problems.

Theorem 5.4. If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

Remark 5.5. This change in the limits of integration is annoying and somewhat unnecessary right now (so long as you change your variable back) however in Calc II once you start doing trig substitutions it becomes extremely useful. WeBWorK will force you to practice changing the limits of integration so we will practice here too.

Example 5.6. Evaluate the following definite integrals by changing the limits of integration appropriately:

(a) $\displaystyle\int_0^1 \sqrt[3]{1+7x}\, dx$

(b) $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\, dx$

5.3 VIDEO - Additional Properties

Objective(s):

- Introduce properties of definite integrals and symmetric functions to create a few more theorems.
- Use them!

Definition(s) 5.7. Recall again

(a) f is called __________ if $f(-x) =$ ________.

(b) f is called _________ if $f(-x) =$ ___________.

Theorem 5.8. Suppose f is continuous on $[-a, a]$ then:

(a) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\ dx =$

(b) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\ dx =$

Example 5.9. Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^2(x)\sec^2(x)\ dx$

Example 5.10. Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} (1 + x + x^2 \tan x)\ dx$

1 Area Between Curves

1.1 VIDEO - X Direction

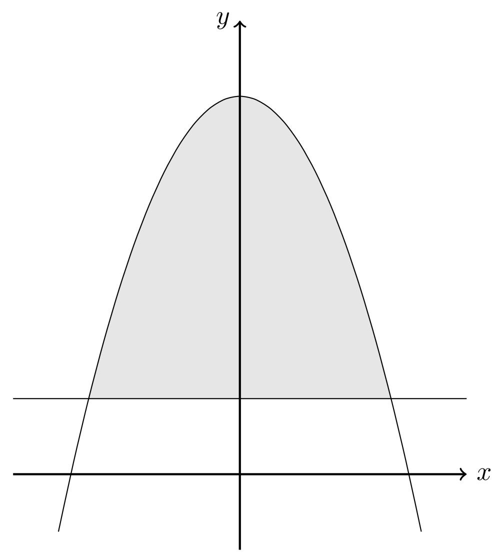
Objective(s):

- Express the area bounded by two curves as a definite integral and evaluate.
- Deal with more difficult situations like when graphs cross.

Theorem 1.1. If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region in between the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

Area =

Example 1.2. Find the area bounded between $y = 1$ and $y = 5 - x^2$.



Remark 1.3. This theorem only applies if $f(x) \geq g(x)$ on $[a, b]$. In the more general case (where the graphs cross), we can use the following theorem.

Theorem 1.4. If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$, then the area of the region in between the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

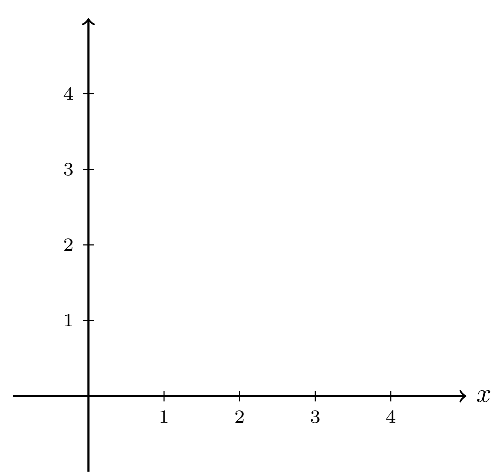
Area =

Example 1.5. Find the area between $y = x^2$ and $y = 3 - 2x$ on the interval $[0, 3]$.

Hint: Here's what this situation looks like `https://www.desmos.com/calculator/tytqdpfng2`

One more thing that sometimes comes up is that the function can switch on us.

Example 1.6. Sketch the lines $y = x$, $y = 4 - x$ and $y = 4 - \frac{x}{3}$. Setup a definite integral or integrals that represents the area bounded between them



1.2 VIDEO - Y Direction

Objective(s):

- Identify when it is advantageous to integrate with respect to y instead of x.
- Get a bit more practice.

Remark 1.7. If it is more convenient, you can think of x as a function of y, and integrate with respect to y. For example, to find the area between the graphs of $x = f(y)$ and $x = g(y)$ between $y = a$ and $y = b$, compute

Area =

Example 1.8. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Take a look at this region via: `https://www.desmos.com/calculator/ynjlvejtqf`

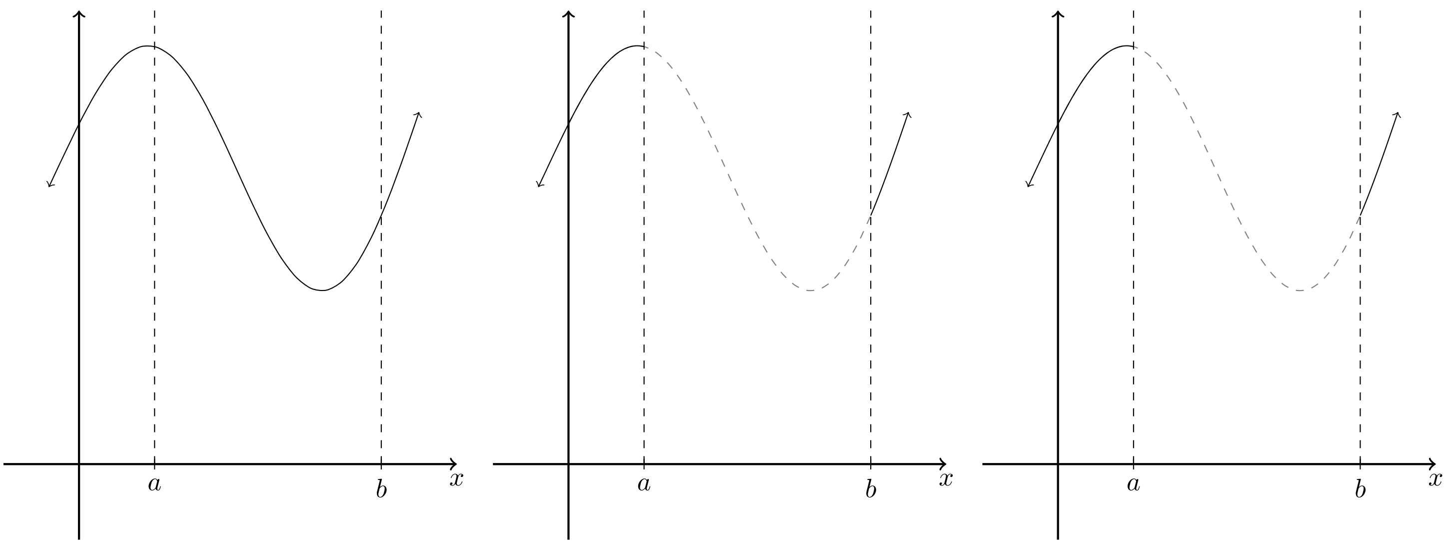
5 Average Value of a Function

5.1 VIDEO - We all love a good average

Objective(s):

- Visualize the average value of a function on an interval.
- Calculate the average value of a function over an interval.
- Relate the average value of a function to averages of finite sets.

This is how I visualize the average value of a function



Definition(s) 5.1. The ________________________ of a function $f(x)$ on the interval $[a, b]$ is defined to be:

$$f_{\text{ave}} =$$

Example 5.2. Find the average value of the following functions on the given interval:

(a) $f(x) = \sqrt{x}$ on $[0, 4]$

(b) $f(x) = \sin(x)$ on $[0, \pi]$

Definition(s) 5.3. Given a set of n values $\{y_1, y_2, \ldots, y_n\}$ the _____________ value, or __________, of the set is denoted ___ and is given by:

Remark 5.4. Break up $[a, b]$ into n subintervals and see how the average of f over a finite set is related to our definition.