Learning Goals

As discussed in class last week, integration really is a process of divide-and-conquer: you wish to compute a complicated quantity, and to do so, you first chop it up into many small pieces, each of which easier to compute, then you add everything back together.

Exact integration (or, finding the antiderivative) requires thinking about the limiting case where the division step divides into *finitely many* pieces each of which are in fact *infinitesimally small*. In most practices, if you just want a "good enough approximation" (which would almost always be the case in science and engineering applications), you can get away with just *very many* pieces each of which are *very small*. This is what we will explore in this lab.

Integrals versus Sums

First let us revisit the idea that the definite integral of a function is defined via a limiting procedure. Given a function $f(x)$, we can draw below it a number of rectangles and compute the sums of the areas of the rectangles.

This is a tedious calculation to do by hand, but we will be using `MATLAB` to do the work for us and help visualize our setup. This Pre-Lab will walk you step by step through the code we'll use, and compare it to the calculations we're familiar with.

Let's start with an example:

Take our function to be $f(x) = \sin(x^2) + 1$, and our domain to be the interval $[0, 3]$. Using the following code, we are able to plot the graph of the function.

```
f = @(x) sin(x.^2) + 1 % Define our function
a = 0 % Left endpoint
b = 3 % Right endpoint
fplot(f, [a, b]) % Plot the functions
hold on % Get ready to plot more
```

We will now draw N rectangles below the graph to show the areas we add together to approximate the integral. Each of these rectangles will touch the graph with their **top left** corners.

In this case, we will set $N = 6$, but with the magic of `MATLAB`, in the actual lab document, we are able to change N on the fly with a slider. The code to plot these rectangles is given below with explanations in the comments.

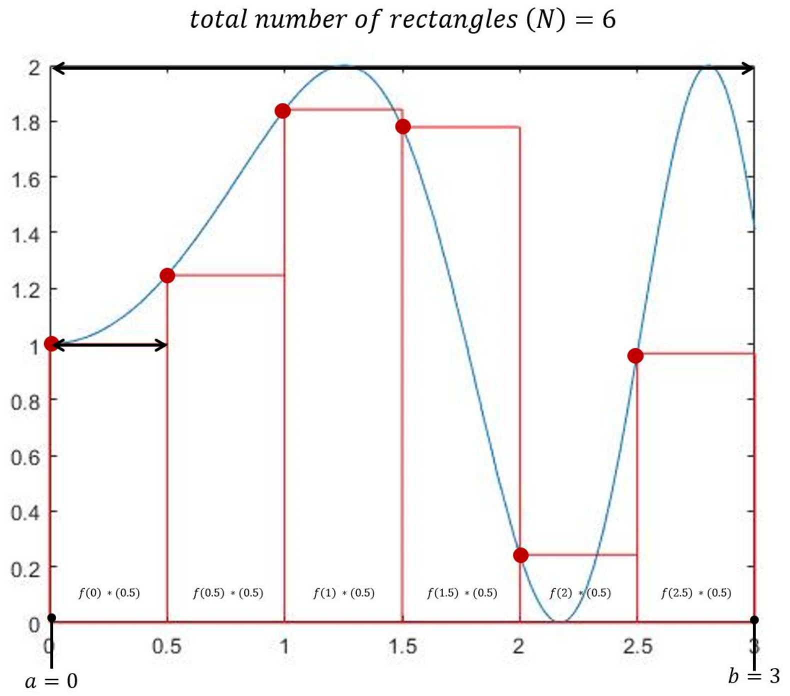
```
N = 6 % Number of divisions

% If we divide the interval [a, b] into N pieces,
% each piece would have width
width = (b - a) / N

% The command below says to look at all values of x,
% starting from a, increasing by the variable 'width'
% each time, until the final value where x = b - width
for x = a:width:(b - width)
    % For each of the x values, we draw a rectangle
    % with the lower left corner at coordinate (x, 0),
    % width the variable 'width', and height the value f(x).
    rectangle('Position', [x 0 width f(x)], 'EdgeColor', 'r')
end
hold off % Done plotting
```

Computing area under curve

As seen from the figure below, with the aid of these rectangles, we can compute an approximation to the area under the curve for the function $f(x)$.



To compute the corresponding Riemann sum, we can use the code

```
approximated_integral = sum(f(a:width:(b-width)) .* width)
```

Let's explain the pieces in this command a little, working from the inside out:

- Remember from the rectangle plotting above, the command `a:width:(b - width)` is the list of x values starting from `a`, ending at `b - width`, spaced `width` apart.
- The command `f(a:width:(b-width))` plugs this list of x values into the function $f(x)$, producing the corresponding list of $f(x)$ values.
- We then multiply these $f(x)$ values by `width` and add them all up.

So for our example with $N = 6$, the this command computes

$$f(0) \cdot 0.5 + f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5 + f(2) \cdot 0.5 + f(2.5) \cdot 0.5$$

which we can also write, using Sigma notation, as

$$\sum_{j=0}^{5} f(0 + j \cdot \Delta x) \cdot \Delta x$$

where the width $\Delta x = (3 - 0)/6 = 0.5$. Notice that this looks almost exactly the same as our `MATLAB` code calculating our approximate integral!

More generally, for arbitrary values of N, the number of divisions, the area can be written as

$$\sum_{j=0}^{N-1} f(0 + j \cdot (3/N)) \cdot (3/N)$$

And as a reminder, the integral as described in Calculus I is the limiting value as the number of divisions go to infinity

$$\int_a^b f(x)dx = \lim_{N\to\infty} \left[\sum_{j=0}^{N-1} f\left(a + j \cdot \frac{b-a}{N}\right) \cdot \frac{b-a}{N} \right].$$

In the main lab, we'll continue to play with these ideas, linking them with `MATLAB` to do the heavy computational lifting for us.