Learning Goals

Since the earliest implementations of mechanical calculators, to modern development of supercomputers, these devices are designed to be extremely efficient in performing the fundamental arithmetic operations of **addition**, **subtraction**, **multiplication**, and **division**.

Evaluations of other functions, however, are not built into the hardware, and must be replicated in software.

Today's lab aims at answering the question: how does a computer/electronic calculator evaluate complicated functions like sin, cos, exp and ln? The answer resides in one of the highlights of Calculus II: the Taylor polynomial.

You will learn how **Taylor polynomials** allow us to efficiently create numerical approximations, as well as the central role played by the corresponding **radius/interval of convergence**. Additionally, you will learn how to manipulate Taylor polynomials and use comparison tests in order to bound the error in a polynomial approximation to a function.

For concreteness, we will focus on the **natural logarithm** function ln.

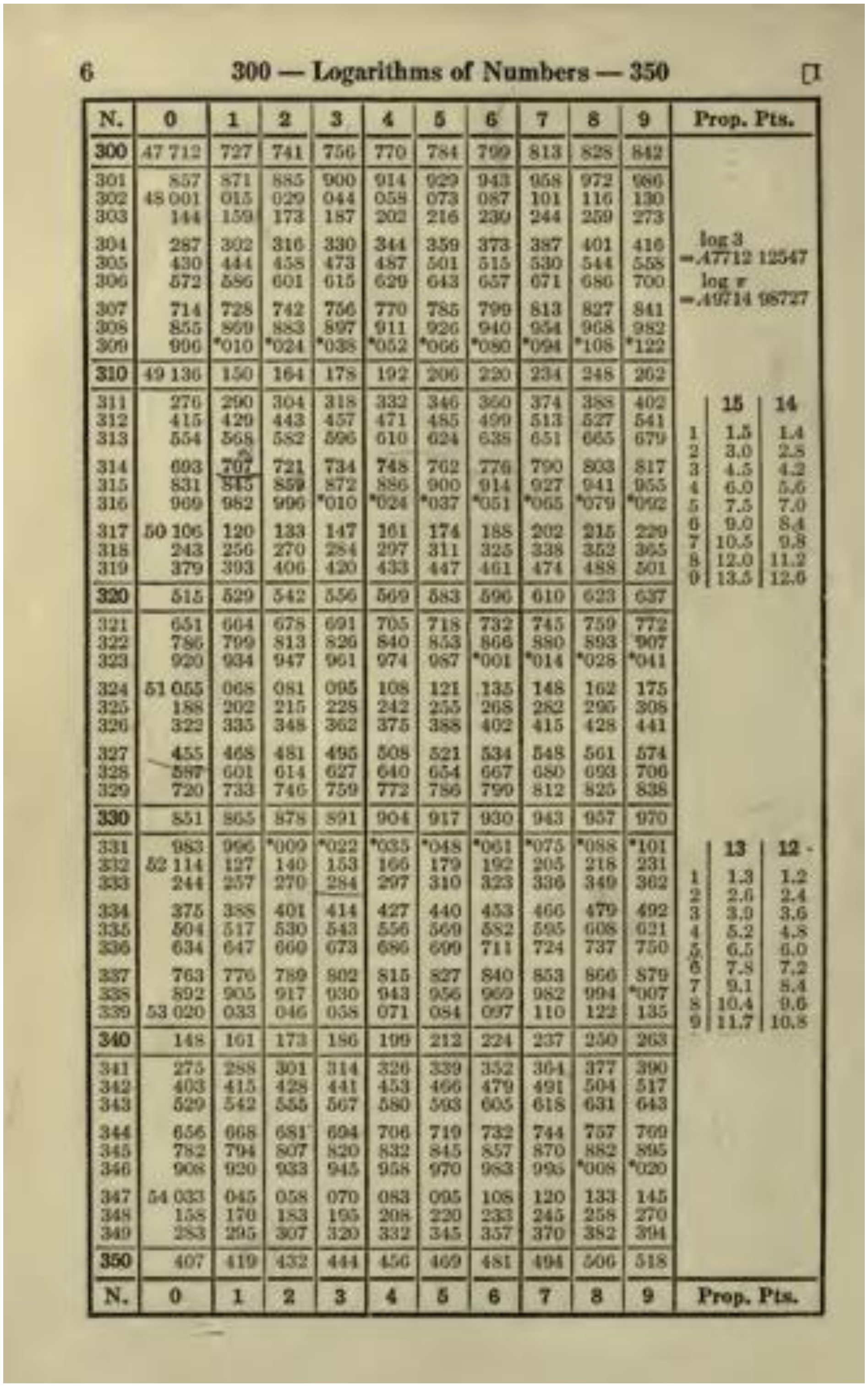
Option 1: A Look-up table

Before the advent of modern electronic calculators, when one needs to use logarithms in practice, one consults a table of logarithms. An excerpt of one such table is shown in Figure 1. The values shown in the table are pre-computed by hand, with the aid of a mechanical calculator.

Such a table can be easily reproduced in a computer, especially since computers store numbers with finite *precision* (that is, only using finitely many digits), meaning that there are only finitely many entries needed in our table.

Except that for modern uses, numbers are represented using 64 bits each. A look-up table covering all valid 64 bit floating point numbers will, in principle, take storage space between 100 petabyte and 1 zetabyte (the latter being 1 billion terabytes). Even for use with a simple 8-digit electronic calculator, a corresponding look-up table will take about 1 gigabyte of storage.

So option 1 is not very practical.



6 **300 — Logarithms of Numbers — 350**

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 3 = .47712 12547

log π = .49714 98727

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

Figure 1: Logarithm table from Logarithmic and trigonometric tables by E.R. Hedrick. The Macmillan Company (1920). (Work in public domain, image extracted from archive.org.)

Option 2: Taylor polynomials

A second option is to use Taylor polynomials. Remember from class that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

with radius of convergence 1.

While it is not really feasible to sum the whole infinite series (which will require adding infinitely many numbers together), we can hope that the Maclaurin polynomial of sufficiently high degree will give a good enough approximation to the actual function for the values of x where the series converges.

Bounding the remainder

Since the radius of convergence of the aforementioned Macularin series is 1, we know that the sum converges to the value of $\ln(1+x)$ for any value of x in the open interval $(-1,1)$. Additionally, it is possible that the series converges at the endpoints -1 and 1, but this would need to be checked separately.

To stay away from the potentially dangerous endpoints, let's consider the difference between the $\ln(1+x)$ and just a few terms of the Maclaurin series on a large chunk of the interval, $[-0.9, 0.9]$. We'll first consider what happens when we use the fifth degree Maclaurin polynomial.

The difference between this fifth degree Maclaurin polynomial and the full Maclaurin series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - \sum_{n=1}^{5} \frac{(-1)^{n+1} x^n}{n} = \sum_{n=6}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Now, we only care about how big this difference is in magnitude, so we will take the absolute value. However, the absolute value of an entire series can be difficult to work with. Instead, we will make use of **absolute convergence** get an upper bound involving the absolute value of each term.

From **absolute convergence** we know that the absolute value of a series is bounded by its corresponding series of absolute values:

$$|\sum a_n| \leq \sum |a_n|$$

So we see that the magnitude of the difference between the 5th degree Maclaurin polynomial and the full Maclaurin series is at most

$$\sum_{n=6}^{\infty} \frac{|x|^n}{n}$$

But now this series doesn't help us out very much, since we'd like something that we can actually sum up, like a geometric series. For this, we use the **direct comparison test**.

From the **direct comparison test**, we know that if two series a_n and b_n have positive terms, and $a_n < b_n$, then their sums satisfy

$$\sum a_n < \sum b_n$$

We can apply this to to our difference, and using that $|x|^n/n \leq x^n$, obtain that

$$\sum_{n=6}^{\infty} \frac{|x|^n}{n} < \sum_{n=6}^{\infty} |x|^n$$

So we reached our conclusion that the difference between the fifth degree Maclaurin polynomial for $\ln(1+x)$ and the full Maclaurin series is at most

$$\sum_{n=6}^{\infty} |x|^n$$

Now, this quantity is a geometric series, so we can sum it and see how large this sum can get on the interval $[-0.9, 0.9]$. This will be the starting point for the lab, where you will use techniques like these to investigate efficient ways to calculate the specific value $\ln(2)$ with Taylor polynomials.