Companion Lecture Notes for MSU's MTH 133

Compiled by: Ryan Maccombs

Contents

Chapter 5

Applications of Integration

5.2 Volumes

Section Objective(s):

- Use integrals to find the volume of a 3D solid
- Identify whether to use the disk, washer, or another method to find volume

Example 5.2.1. Let R be the region bounded by $y = x^2$, $x = 1$, and the x-axis. Find the area of R by integrating

(a) with respect to x.

(b) with respect to y.

Example 5.2.2. Let R be the region bounded by $y = x^2$, $x = 1$, and the x-axis. Find the volume of the solid generated when R is rotated around the x-axis.

Definition(s) 5.2.3. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the ______________ of S is

Definition(s) 5.2.4. The solid in the previous example is called a ________________________________ because it is obtained by revolving a region about a line.

Theorem 5.2.5. In general we can calculate the volume of a solid of revolution by using the basic defining formula

$$V = \qquad\qquad\qquad \text{or} \qquad\qquad V =$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

- If the cross-section is a disk, we find the radius of the disk(in terms of x or y) and use

$$A =$$

- If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

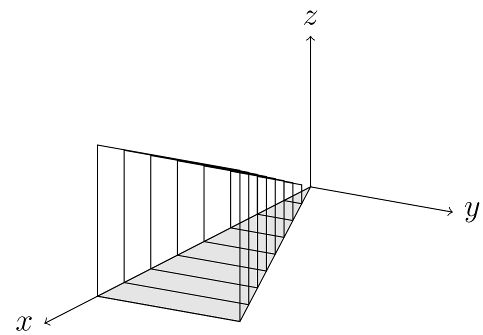
$$A =$$

Example 5.2.6. Let R be the region bounded by $y = x^2$, $x = 1$, and the x-axis. Find the volume of the solid generated when R is rotated around the y-axis.

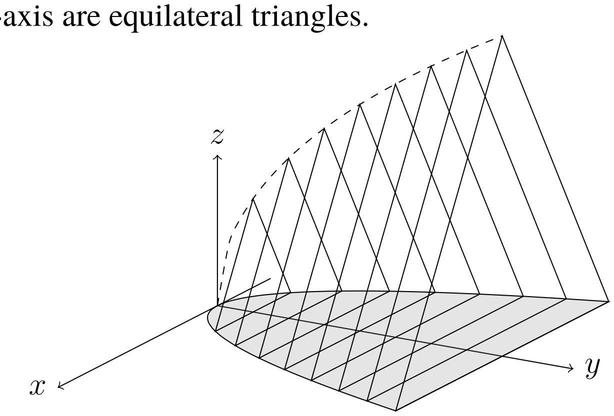
Example 5.2.7. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 0$, $y = 2$, and $x = 1$ about the line $y = 3$. Draw a picture of the volume you are finding.

Example 5.2.8. Consider the region of the xy-plane bounded by $y = 0$, $x = 0$ and $y = 1 - x$. Find the volume of the solid generated by revolving this region about the line $x = 2$.

Example 5.2.9. Consider the solid with triangular base formed by $y = x/2$, $y = 0$ and $x = 4$, for which parallel cross-sections perpendicular to the base and x-axis are squares. Find the volume of such a solid (shown below).



Example 5.2.10. Find the volume of the solid whose base B is the region bounded by the parabola $y = x^2$ and $y = 1$ and whose cross sections perpendicular to the y-axis are equilateral triangles.



5.4 Work

Section Objective(s):

- Use integrals to find work
- Determine the equation for force in a given situation

Definition(s) 5.4.1. We define the _________ **done in moving an object from** __________ as:

Example 5.4.2. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Example 5.4.3. A 10 ft cable weighing 50 lbs and a bucket of water weighing 60 lbs is hanging down in a well. How much work must be done to lift bucket and cable to the top of the well?

Definition(s) 5.4.4. ______________________ states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x. That is: _________________ where ___ is a ____________ constant.

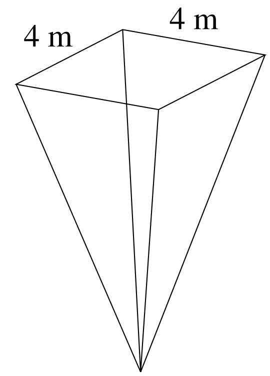
Example 5.4.5. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 14 cm to 16 cm?

Example 5.4.6. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with oil weighing 30 pounds per cubic foot to a height of 8 m. Find the work required to empty the tank by pumping all of the oil to the top of the tank.

Theorem 5.4.7. If σ is the density of the liquid, $d(y)$ is the distance to the where the liquid is being pumped, $A(y)$ is the area of a cross selection of liquid perpendicular to the y-axis and the liquid exists between $y = a$ and $y = b$ then

$$\text{Work} =$$

Example 5.4.8. A tank has the shape of an inverted right rectangular cone with height 8 m and base rectangle $4\text{m} \times 4\text{m}$ shown below. It is filled with oil weighing 20 N/m^3. Find the work required to pump the oil to a spot 3 meters above the tank.



Chapter 6

Inverse Functions:

Exponential, Logarithmic, and Inverse Trigonometric Functions

Forest for the Trees

Months from now you will look back and long for the simpler times of Chapter 6. In Chapter 6 we add a few more functions to our calculus tool kit including how to differentiate and integrate functions related to:

- Exponentials, Logarithms, Inverse Trigonometrics, and Hyperbolics

In addition, we will learn how to differentiate in style by using a technique called logarithmic differentiation. We will also study how to defeat more interesting limits using L'Hospital's Rule. By the end of Chapter 6 you will be able to solve problems such as:

Example: Find the derivatives of:

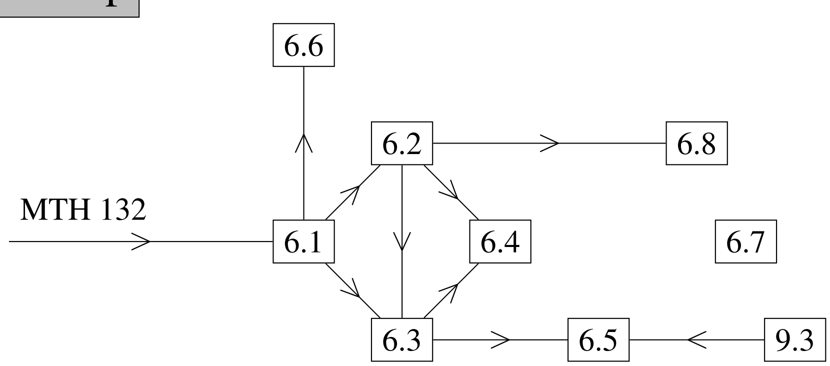
$$(a) \quad f(x) = \ln x + \log_5 x \qquad\qquad (b) \quad g(x) = e^x + 2^x$$
$$(c) \quad h(x) = (2x)^{3x} \qquad\qquad (d) \quad h(x) = \sin^{-1}(x) + \tanh(x)$$

Example: Bill Bourbon was murdered in Savannah sometime last night. The temperature outside has been a constant 65° F. At 9AM Bill's body temperature was $82°$ F and at 10AM his temperature was $80°$ F. Nathaniel Nutmeg was seen at the bar from 9PM-2AM last night. Could he have committed the murder? (*Assume initial body temperature was* $98.6°$ F.)

Example: Evaluate the following limits:

$$(a) \quad \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) \qquad\qquad (b) \quad \lim_{x\to 0^+} (2x)^{3x}$$
$$(c) \quad \lim_{x\to 0^+} \left(\frac{3}{x}\right)^{5x} \qquad\qquad (d) \quad \lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{x}$$

Chapter 6 Map



6.1 Inverse Functions

Section Objective(s):

- Find the inverse of a function
- Evaluate the derivative of the inverse of a given function

Definition(s) 6.1.1. A function f is called a ________________________________ if it never takes on the same value twice; that is,

Remark 6.1.2. ________________________________: A function is one-to-one if and only if _____ horizontal line intersects its graph ________________________

Definition(s) 6.1.3. Let f be a one-to-one function with domain A and range B. Then its ________________________________ has domain B and range A and is defined by

for any y in B.

Remark 6.1.4. Note that:

domain of f^{-1} = range of f

range of f^{-1} = domain of f

Definition(s) 6.1.5. Cancellation equations:

for every x in A

for every x in B

Theorem 6.1.6. How to Find the Inverse Function of a One-to-One Function f:

Step 1 Write ________________

Step 2 Solve this equation for ___ in terms of ___ (if possible).

Step 3 To express f^{-1} as a function of x, ______________________________. The resulting equation is $y =$ ___________.

Example 6.1.7. Find the inverse function of $f(x) = x^3 + 2$.

Remark 6.1.8. The graph of f^{-1} is obtained by reflecting the fraph of f about the line $y = x$.

Theorem 6.1.9. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} ______________________________.

Theorem 6.1.10. If f is a one-to-one ______________________ function with inverse function f^{-1} and __________________________, then the inverse function is ______________________ at a and

Example 6.1.11. If $f(x) = x^3 + 2$, find $(f^{-1})'(3)$.

Example 6.1.12. Consider the function $f(x) = \dfrac{4x+1}{2x+3}$

(a) Find the inverse of f

(b) Find $(f^{-1})'(1/3)$

6.2 The Natural Logarithmic Function

Section Objective(s):

- Identify properties of the natural logarithmic function.
- Explore the process of logarithmic differentiation

Definition(s) 6.2.1. The ______________________________ is the function defined by

Theorem 6.2.2. Laws of Logarithms If x and y are positive numbers and r is a rational number, then

(a) $\ln(xy) =$

(b) $\ln\left(\dfrac{x}{y}\right) =$

(c) $\ln(x^r) =$

Example 6.2.3. Express $\ln(3) + 4\ln(4x)$ as a single logarithm.

Remark 6.2.4.

$$\frac{d}{dx}(\ln x) =$$

Definition(s) 6.2.5. __ is the number such that _______________.

Remark 6.2.6. In general, if we combine Remark 6.2.2 with the Chain Rule, we get

Example 6.2.7. Find the derivative $\dfrac{d}{dx}\ln(\sin(3x^2))$.

Remark 6.2.8.

$$\frac{d}{dx}(\ln|x|) = \qquad\qquad \int \frac{1}{x}dx =$$

$$\int \tan\ x\ dx =$$

Theorem 6.2.9. Steps in Logarithmic Differentiation

1. Take ______________________________ of both sides of an equation $y = f(x)$ and use the _________________________________ to simplify.

2. _____________________ implicitly with respect to ___.

3. Solve the resulting equation for ____.

Example 6.2.10. Compute the derivative for the following function: $y = \dfrac{(\sin^2 x)(x+3)^4}{(5x-8)^{10}}$

Example 6.2.11. Find the derivative of $y = t^2 + 3\ln(5\ln(t))$

Example 6.2.12. Evaluate the integral $\int_9^{10} \frac{1}{t \ln t}\, dt$

6.3 The Natural Exponential Function

Section Objective(s):

- Identify properties of the natural exponential function.
- Practice differentiation/integration of the natural exponential function.

Definition(s) 6.3.1. The inverse of the ____
is the denoted by ____________. We define

$$\iff$$

Cancellation equations:

and

Example 6.3.2. Solve the equation $2e^{2-x} = 30$.

Theorem 6.3.3. Properties of the Natural Exponential Function The exponential function $f(x) =$ ____ is an increasing continuous function with domain $\mathbb{R}$ and range $(0,\infty)$. Thus ____________ for all x. Also

So the x-axis is a horizontal __________________ of $f(x) = e^x$.

Definition(s) 6.3.4. Laws of Exponents If x and y are real numbers and r is rational, then

(a) $e^{x+y} =$

(b) $e^{x-y} =$

(c) $(e^x)^r =$

Remark 6.3.5. The natural exponential function has the remarkable property that it is its own derivative.

In general, if we combine this with the Chain Rule, we get

$$\frac{d}{dx}(e^u) =$$

Example 6.3.6. Evaluate the derivative $\frac{d}{dx}e^{\sin(3x)}$.

Example 6.3.7. $\int e^x \, dx$

Example 6.3.8. Find the equation of the tangent line to the curve $y = \dfrac{4e^{-x}}{x^2}$ at $(-2, e^2)$.

Example 6.3.9. Evaluate the integral $\displaystyle\int 4e^x\sqrt{3+e^x}\, dx$

6.4 General Logarithmic and Exponential Functions

Section Objective(s):

- Explore the properties and calculus of general exponential functions.
- Explore the properties of the general logarithmic function.

Definition(s) 6.4.1. We define

$$a^x =$$

Theorem 6.4.2. The general ________________________________ follow from this together with the laws for e^x. If x and y are real numbers and $a, b > 0$, then

(a) $a^{x+y} =$

(b) $a^{x-y} =$

(c) $(a^r)^x =$

(d) $(ab)^x =$

Theorem 6.4.3.

(a) $\dfrac{d}{dx}(a^x) =$

(b) $\int a^x dx =$ when $a \neq 1$

Example 6.4.4. Evaluate $\int 4^{-x} dx$.

Theorem 6.4.5. The Power Rule If n is any real number and $f(x) = x^n$, then

Definition(s) 6.4.6. If $a > 0$ and $a \neq 1$, then $f(x) =$ ____ is a one-to-one function. Its inverse function is called the ____________________ **function with** ___________ and is denoted by _______. Thus

$$\iff$$

Remark 6.4.7. Change of Base Formula For any positive number a with $a \neq 1$, we have

Example 6.4.8. Evaluate the following: $\log_8 256$

Theorem 6.4.9. $\dfrac{d}{dx}(\log_a x) =$

Example 6.4.10. Use logarithmic differentiation to find dy/dx for $y = (1+x)^{1/x}$

Example 6.4.11. Find the derivative of

(a) $y = 4^x + x^4$

(b) $f(x) = 3^{\sin(2x)}$

(c) $g(x) = (2x)^{3x}$ for $x > 0$

Example 6.4.12. Evaluate the integrals

(a) $\displaystyle\int x5^{x^2}\,dx$

(b) $\displaystyle\int 2^{\sin\theta}\cos\theta\,d\theta$

6.5 Exponential Growth and Decay

Section Objective(s):

- Explore different applications of exponential functions.
- Recognize exponentials are the solutions to certain differential equations.

Theorem 6.5.1. The only solutions of the differential equation ____________________ are the exponential functions

$$y(t) =$$

Definition(s) 6.5.2. In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

or

This k is the growth rate divided by the population size; it is called the **relative growth rate**. If the population at time 0 is P_0, then the expression for the population is

$$P(t) =$$

Example 6.5.3. The number of cases of a disease is reduced by 20% each year. If there are 10,000 cases today, how long will it take to reduce the number to 1000?

Definition(s) 6.5.4. Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t

where ___ is a negative constant. The mass decays exponentially:

$$m(t) =$$

Example 6.5.5. Element X is radioactive. 7 days ago my sample of element X weighed 100 grams but today it only weighs 90 grams. How many days until it weighs 45 grams?

Definition(s) 6.5.6. Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay. This can be found by

$$\text{half-life} =$$

Definition(s) 6.5.7. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then

where k is a constant. This can be solved to find

$$T(t) =$$

Example 6.5.8. Bill Bourbon was murdered in Savannah sometime last night. The temperature outside has been a constant 65° F. At 9AM Bill's body temperature was $82°$ F and at 10AM his temperature was $80°$ F. Nathaniel Nutmeg was seen at the bar from 9PM-2AM last night. Could he have committed the murder? (*Assume initial body temperature was* $98.6°$ F.)

9.3 Separable Equations

Section Objective(s):

- Apply techniques to solve separable differential equations.
- Recall how to solve initial value problems.

Definition(s) 9.3.1. A ______________________________ is a first-order differential equation in which the expression for dy/dx can be written in the form:

Equivalently, if ______________ then we could write it as:

where $h(y) = 1/g(y)$.

Technique for solving separable differential equations

1. Rewrite the equation in the differential form:
2. Integrate both sides:
3. Solve for y in terms of x (if possible)

Example 9.3.2. Find the general solution of the differential equation $2yy' - x = 0$

Example 9.3.3. Solve the initial value problems:

(a) $y\dfrac{dy}{dx} - e^x = 0, \quad y(0) = 4.$

(b) $2x\dfrac{dy}{dx} - \ln x^2 = 0, \quad y(1) = 2.$

(c) $\dfrac{dT}{dt} = -k(T - T_s), \quad T(0) = T_0$ (*to verify the formula for Newton's Law of Cooling*)

6.6 Inverse Trigonometric Functions

Section Objective(s):

- Define the inverse trigonometric functions
- Understand the calculus of the inverse trigonometric functions

Definition(s) 6.6.1. The ______________________________, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

$$\iff \qquad\qquad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

The cancellation equations for inverse functions become, in this case,

$$\text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\text{for } -1 \leq x \leq 1$$

and its derivative is given by

$$-1 < x < 1$$

Example 6.6.2. Evaluate $\dfrac{d}{dx}\sin^{-1}(\ln(3x^2))$

Definition(s) 6.6.3. The **inverse** ____________ **function** is handled similarly.

$$\cos^{-1} x = y \quad \Longleftrightarrow \quad \text{and} \quad 0 \le y \le \pi$$

The cancellation equations for inverse functions become, in this case,

for $0 \le x \le \pi$

for $-1 \le x \le 1$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$, and its derivative is given by

$$-1 < x < 1$$

Definition(s) 6.6.4. The **inverse** ______________ **function** is defined

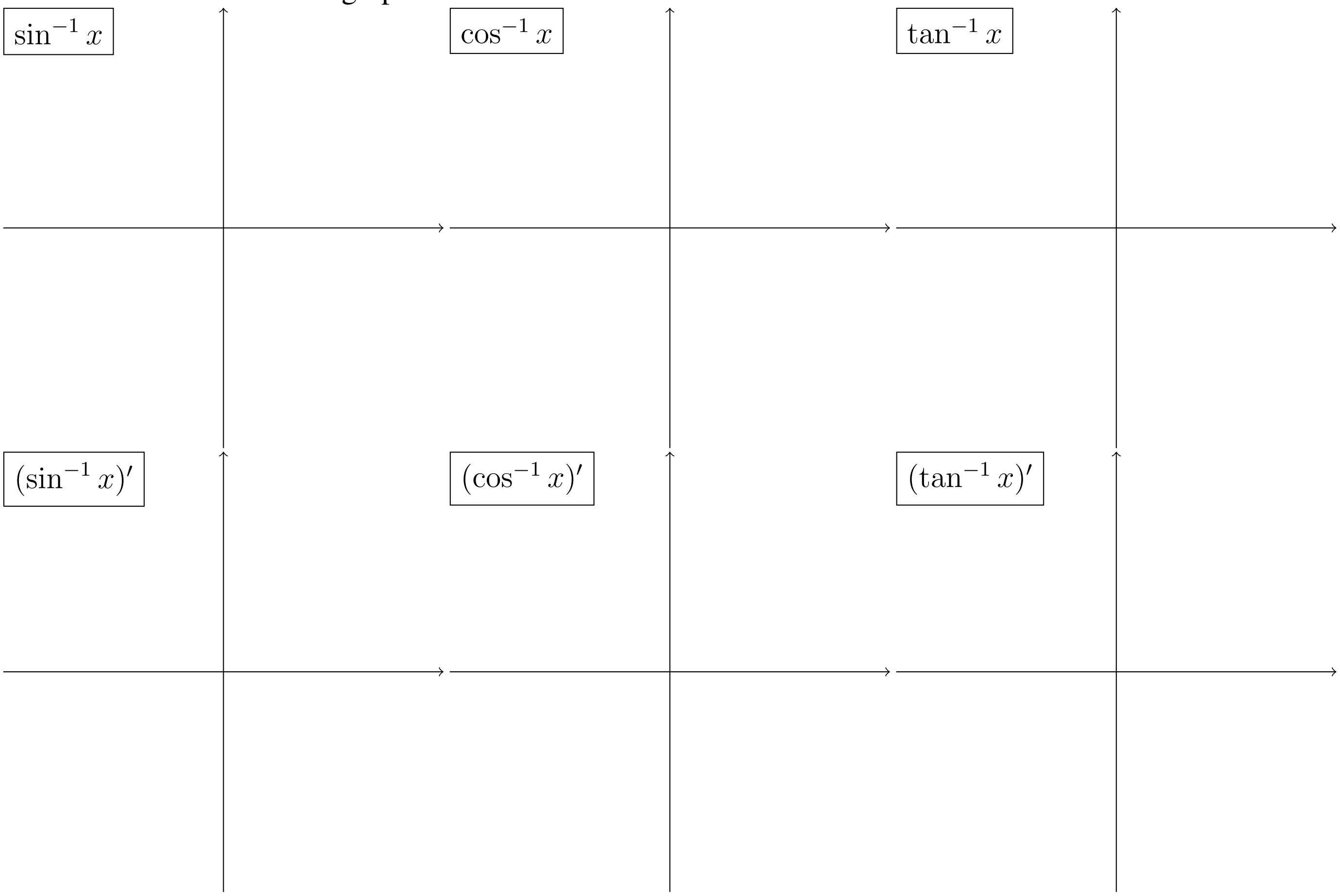
$$\tan^{-1} x = y \quad \Longleftrightarrow \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The lines _______________ and __________________ are horizontal asymptotes of the graph of $\tan^{-1}$.

Its derivative is given by

$$\frac{d}{dx}(\tan^{-1} x) =$$

Remark 6.6.5. Pictures of graphs and derivatives



Example 6.6.6. Evaluate $\dfrac{d}{dx} \cos^{-1}(5^x)$

Example 6.6.7. Evaluate $\displaystyle\int \frac{3}{(2x)^2+1}\,dx$

Theorem 6.6.8. The remaining inverse trigonometric functions are not used as frequently and are summarized here.

$$y = \csc^{-1}(x) \quad (|x| \geq 1) \qquad \Longleftrightarrow$$

$$y = \sec^{-1}(x) \quad (|x| \geq 1) \qquad \Longleftrightarrow$$

$$y = \cot^{-1}(x) \quad (x \in \mathbb{R}) \qquad \Longleftrightarrow$$

Theorem 6.6.9. Table of Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \qquad \frac{d}{dx}(\csc^{-1} x) =$$

$$\frac{d}{dx}(\cos^{-1} x) = \qquad \frac{d}{dx}(\sec^{-1} x) =$$

$$\frac{d}{dx}(\tan^{-1} x) = \qquad \frac{d}{dx}(\cot^{-1} x) =$$

Theorem 6.6.10.

$$\int \frac{1}{\sqrt{1-x^2}}dx = \qquad \int \frac{1}{x^2+1}dx =$$

Example 6.6.11. $\displaystyle\int \frac{1}{x^2+a^2}dx$

Example 6.6.12. Evaluate: $\displaystyle\int \frac{dx}{x^2+2x+5}$. (***Hint**: complete the square*)

Example 6.6.13. Evaluate: $\displaystyle\int \frac{x+1}{1+x^2}\,dx$.

6.7 Hyperbolic Functions

Section Objective(s):

- Introduce and define the hyperbolic functions
- Examine the properties of the hyperbolic functions

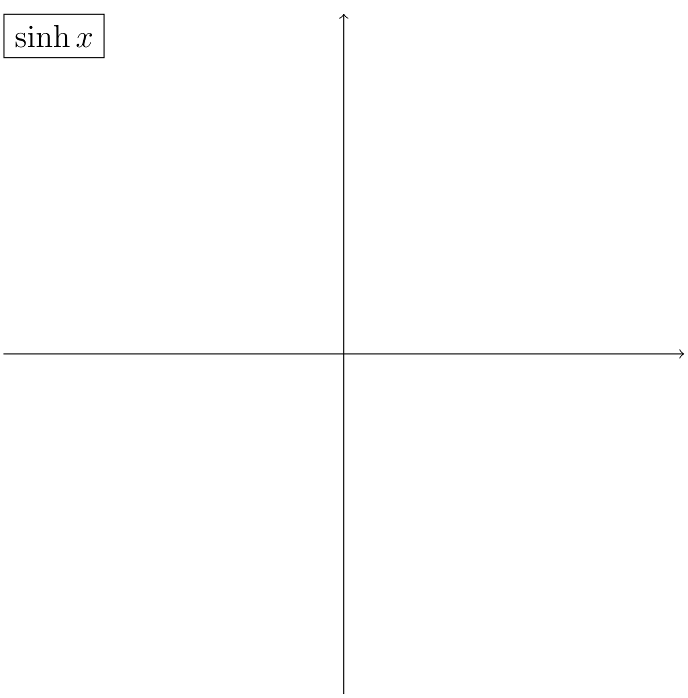
Definition(s) 6.7.1. Definition of the Hyperbolic Functions

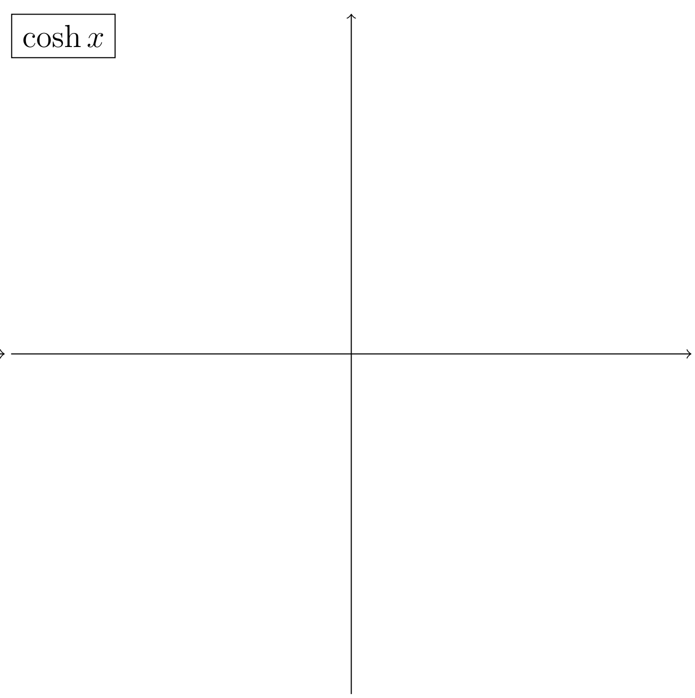
$\sinh(x) =$ $\qquad\qquad$ $\mathrm{csch}(x) =$

$\cosh(x) =$ $\qquad\qquad$ $\mathrm{sech}(x) =$

$\tanh(x) =$ $\qquad\qquad$ $\coth(x) =$

Remark 6.7.2. Graphs of hyperbolic sine and cosine



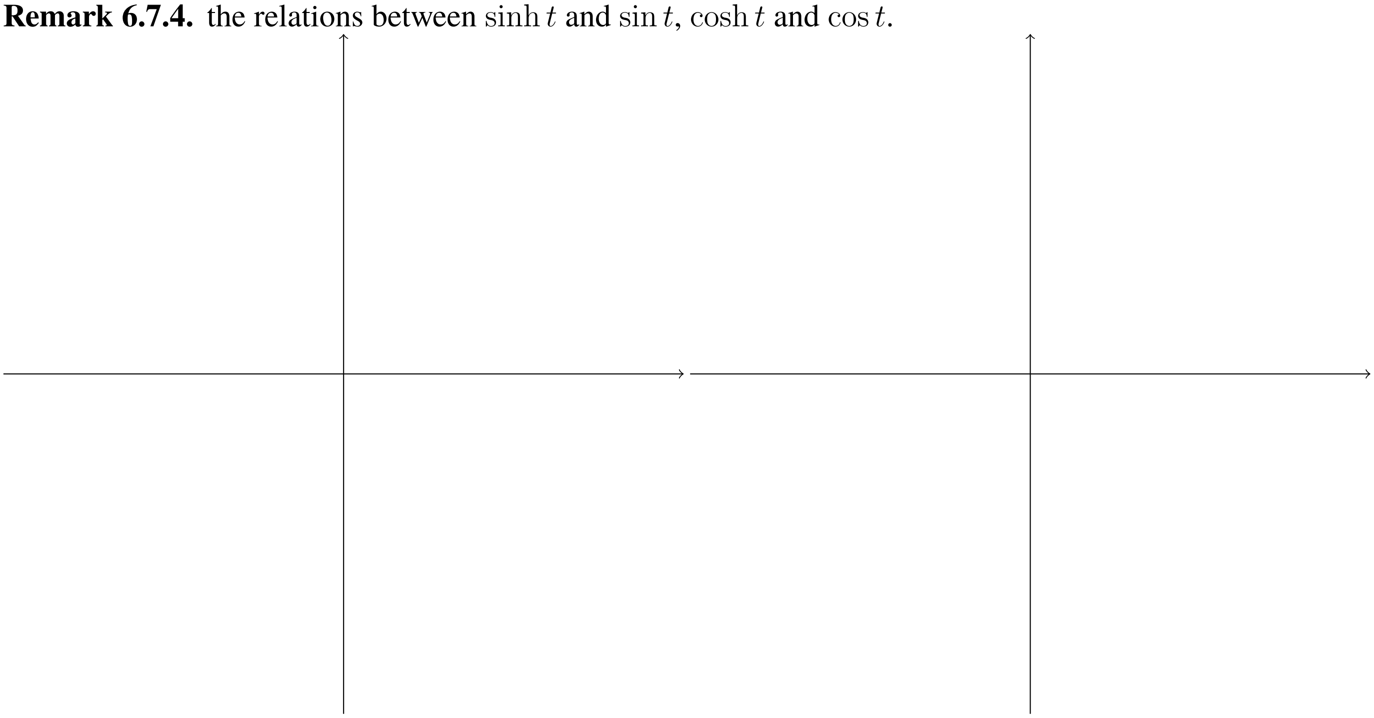


Theorem 6.7.3. Hyperbolic Identities

$$\sinh(-x) = \qquad\qquad \cosh(-x) =$$

$$\cosh^2 x - \sinh^2 x = \qquad\qquad 1 - \tanh^2 x =$$

Remark 6.7.4. the relations between $\sinh t$ and $\sin t$, $\cosh t$ and $\cos t$.



Theorem 6.7.5. Table of Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sinh x) = \qquad\qquad \frac{d}{dx}(\text{csch}\, x) =$$

$$\frac{d}{dx}(\cosh x) = \qquad\qquad \frac{d}{dx}(\text{sech}\, x) =$$

$$\frac{d}{dx}(\tanh x) = \qquad\qquad \frac{d}{dx}(\coth x) =$$

Example 6.7.6. Evaluate the following derivatives:

(a) $\dfrac{d}{dx}\tanh(\sqrt{1+x^2})$

(b) $\dfrac{d}{d\theta}\sinh(\ln(\cosh\theta))$

Example 6.7.7. Evaluate the limit: $\lim\limits_{x\to\infty}\dfrac{\sinh(2x)}{\cosh(3x)}$.

Example 6.7.8. Evaluate the following integrals:

(a) $\displaystyle\int \sinh(3x+1)\,dx$

(b) $\displaystyle\int \tanh(4x)\,dx$

(c) $\displaystyle\int_0^1 t^3(\cosh^2(5t) - \sinh^2(5t))\,dt$

6.8 Indeterminate Forms and L'Hospital's Rule

Section Objective(s):

- Explore indeterminate forms of limits
- Understand and apply L'Hospital's Rule

Definition(s) 6.8.1. If we have a limit of the form

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then the limit may or may not exist and is called an **indeterminate form** ________________.

If we have a limit of the form

where both $f(x) \to \infty(\text{or } -\infty)$ and $g(x) \to \infty(\text{or } -\infty)$ as $x \to a$, then the limit may or may not exist and is called an **indeterminate form of** ______________.

Theorem 6.8.2. L'Hospital's Rule Suppose f and g are differentiable and and $g'(x) \neq 0$ on an open interval *I* that contains a (except possibly at a). Suppose that

and

or that

$=$ and

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} =$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 6.8.3. Use L'Hospital's Rule to calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

Remark 6.8.4. There are additional indeterminate forms such as:

We will demonstrate how to utilize L'Hospital's Rule to solve these.

Calculus 1 Theorem: if $f(x)$ is continuous then:

if the limit exists.

Example 6.8.5. Use L'Hospital's Rule to calculate the following limits:

(a) $\lim_{x\to\infty} \frac{1}{x} \ln x$

(b) $\lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

(c) $\lim_{x\to 0^+} (2x)^{3x}$

Example 6.8.6. Use L'Hospital's Rule to calculate the following limits:

(a) $\lim\limits_{x\to 0^+} \left(\dfrac{3}{x}\right)^{5x}$

(b) $\lim\limits_{x\to \infty} \left(\dfrac{3}{x}\right)^{5x}$

(c) $\lim\limits_{x\to \infty} \left(1+\dfrac{2}{x}\right)^{x}$

Chapter 7

Techniques of Integration

Forest for the Trees

Chapter 7 is all about integration. We will learn quite a few new tricks such as Integration by Parts, Trigonometric Substitution, Partial Fractions, and Improper Integrals. In fact we will learn so many tricks/techniques that it will be hard to keep them all straight. Because of this we will end the Chapter with 7.5, Strategy for Integration, which helps us strategise what we should when we encounter a new integral.

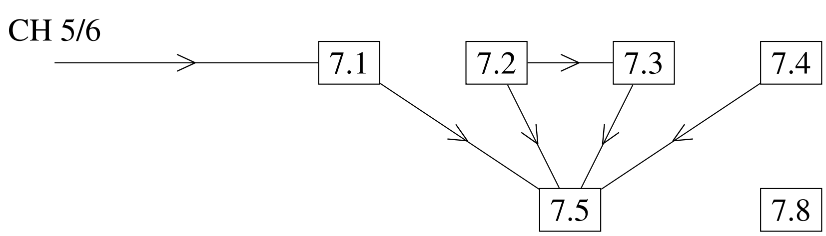
By the end of Chapter 7 you should be able to solve problems such as:

Example : Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}$

Example : Evaluate $\displaystyle\int e^x \cosh 2x \; dx$

Example : Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2} \; dx$

Chapter 7 Map



7.1 Integration by Parts

Section Objective(s):

- Learn the method of integration by parts
- Apply integration by parts to a variety of problems

Theorem 7.1.1. Formula for integration by parts

$$\int f(x)g'(x)dx =$$

Let $u = f(x)$ and $v = g(x)$. By the Substitution Rule, the formula becomes

Remark 7.1.2. We can evaluate definite integrals by parts:

$$\int_a^b f(x)g'(x)dx =$$

Example 7.1.3. Evaluate $\int x \cos x \, dx$

Example 7.1.4. Evaluate $\int x^2 e^{3x} dx$

Example 7.1.5. Evaluate $\int_1^2 \ln x \, dx$

Example 7.1.6. Evaluate $\int \tan^{-1}(5x) \, dx$

Example 7.1.7. Evaluate $\int e^x \cosh 2x \, dx$

7.2 Trigonometric Integrals

Section Objective(s):

- Identify strategies for integrating certain combinations of trigonometric functions
- Integrate some integrals!

Theorem 7.2.1. Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx =$$

Then substitute $u =$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx =$$

Then substitute $u =$

Example 7.2.2. Evaluate $\int \sin^4 x \cos^3 x \, dx$

Theorem 7.2.1 Continued:

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \qquad\qquad\qquad\qquad \cos^2 x =$$

It is sometimes helpful to use the identity

$$\sin x \cos x =$$

Example 7.2.3. Evaluate $\int \sin^2 x \cos^2 x dx$

Theorem 7.2.4. Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x dx =$$

Then substitute $u =$

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x dx =$$

Then substitute $u =$

Example 7.2.5. Evaluate $\int \tan^4 x \sec^6 x \, dx$

Theorem 7.2.6. We will sometimes need to be able to integrate $\tan x$ or $\sec x$:

- $\displaystyle\int \tan x \, dx =$

- $\displaystyle\int \sec x \, dx =$

Proof:

Example 7.2.7. Evaluate $\int \tan^3 x \, dx$

Example 7.2.8. Evaluate $\int x \sin^3 x \, dx$

Theorem 7.2.9. To evaluate the integrals

(i) $\int \sin(mx)\cos(nx)\,dx$,

(ii) $\int \sin(mx)\sin(nx)\,dx$,

(iii) $\int \cos(mx)\cos(nx)\,dx$,

use the corresponding identities:

(a) $\sin A \cos B =$

(b) $\sin A \sin B =$

(c) $\cos A \cos B =$

Example 7.2.10. Evaluate $\int \sin(2x)\cos(3x)\,dx$

7.3 Trigonometric Substitution

Section Objective(s):

- Understand the trigonometric substitution technique for integrating certain functions
- Identify integrals problems that can be solved with this technique.

Definition(s) 7.3.1. In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule, we obtain

$$\int f(x)dx =$$

This kind of substitution is called ________________________________

Theorem 7.3.2. Table of Trigonometric Substitutions

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x =$ $\quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	
$\sqrt{a^2 + x^2}$	$x =$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	
$\sqrt{x^2 - a^2}$	$x =$ $\quad -0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	

Example 7.3.3. Evaluate $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

Example 7.3.4. Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Example 7.3.5. Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

Example 7.3.6. Evaluate $\int \frac{\sqrt{x^2 - 25}}{x}\, dx$

7.4 Integration of Rational Functions by Partial Fractions

Section Objective(s):

- Understand how to break apart fractions in a new way.
- Recall how to solve systems of equations.
- Learn the partial fraction method of integrating rational functions

Definition(s) 7.4.1. ____________________________ can be used to help integrate many ______________ functions. It essentially "_____________ a common ______________________" in reverse.

Example 7.4.2. Use the fact that $\dfrac{2x+1}{x(x+1)} = \dfrac{1}{x} + \dfrac{1}{1+x}$ to evaluate $\displaystyle\int \frac{2x+1}{x^2+x}\,dx$

Partial Fractions Technique: for the rational function $\dfrac{f(x)}{g(x)}$ where deg(f) <deg(g)

1. Factor the ______________________ into __________ and irreducible ________________ terms.
2. Express the rational function as a sum of **partial fractions** of the form

   or

   where i takes on ______________________ less than or equal to the power of the factor.
3. Solve for the ________________________________ using algebraic techniques for solving ______________ of __________________________________.

Example 7.4.3. Evaluate $\int \frac{5x-3}{x^2-2x-3}dx$

Example 7.4.4. Evaluate $\int \frac{x+4}{(x+1)^2}dx$

Example 7.4.5. Evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

Remark 7.4.6. To apply partial fractions for rational functions $\dfrac{f(x)}{g(x)}$ where ________________________ include the additional step

0. ____________ the rational function using polynomial ______________________ to express it as the sum of a ___________________ and a proper rational function.

Example 7.4.7. Evaluate $\displaystyle\int \frac{x^5+5x^4+7x^3+x^2+x-2}{x^4+x^2}\,dx$

7.8 Improper Integrals

Example 7.8.1. Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$

Section Objective(s):

- Understand the need for improper integrals
- Classify different types of improper integrals
- Identify when improper integrals are needed
- Evaluate some improper integrals

Definition(s) 7.8.2. Improper Integral of Type 1

(a) If $\displaystyle\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

provided this limit exists(as a finite number).

(b) If $\displaystyle\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

provided this limit exists (as a finite number).

The improper integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_{-\infty}^b f(x)\,dx$ are called ____________________ if the corresponding limit exists and _________________ if the limit does not exist.

(c) If both $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

In part (c) any real number a can be used.

Example 7.8.3. Determine if the following integral converges or diverges: $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Theorem 7.8.4. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Proof:

Definition(s) 7.8.5. Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

if this limit exists(as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

if this limit exists(as a finite number).

The improper integral $\displaystyle\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are convergent, then we define

Example 7.8.6. Determine if the following integral converges or diverges: $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Theorem 7.8.7. Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\displaystyle\int_a^{\infty}$ ________ dx is convergent, then $\int_a^{\infty}$ ________ dx is convergent.

(b) If $\displaystyle\int_a^{\infty}$ ________ dx is divergent, then $\int_a^{\infty}$ ________ dx is divergent.

Picture:

Example 7.8.8. Determine if the following integral converges or diverges: $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$

7.5 Strategy for Integration

Section Objective(s):

- Review the various formulas for integration
- Overview a general strategy for integration

Remark 7.5.1. A Four-Step Strategy for Integration

1. ____ **if possible** Sometimes the use of algebraic manipulation or trigonometric identities will make the method of integration obvious.

2. **Look for an obvious** ______________________ Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\,dx$ also occurs, apart from a constant factor.

3. ______________ **the integrand according to its** _________ If steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$ which could include any of the following:

   (a)

   (b)

   (c)

   (d)

4. __________________ If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

Example 7.5.2. Evaluate $\displaystyle\int_0^1 \frac{5x+1}{3x+2}\, dx$

Example 7.5.3. Evaluate $\int x^3 e^{x^2}\, dx$

Example 7.5.4. Evaluate $\int \sqrt{x^2 - 2x + 2}\, dx$

Chapter 8

Further Applications of Integration

8.1 Arc Length

Section Objective(s):

- Understand the formula for calculating arc length

Theorem 8.1.1. The Arc Length Formula If ______________________________ on $[a, b]$, then the length of the curve $y = f(x), a \leq x \leq b$, is

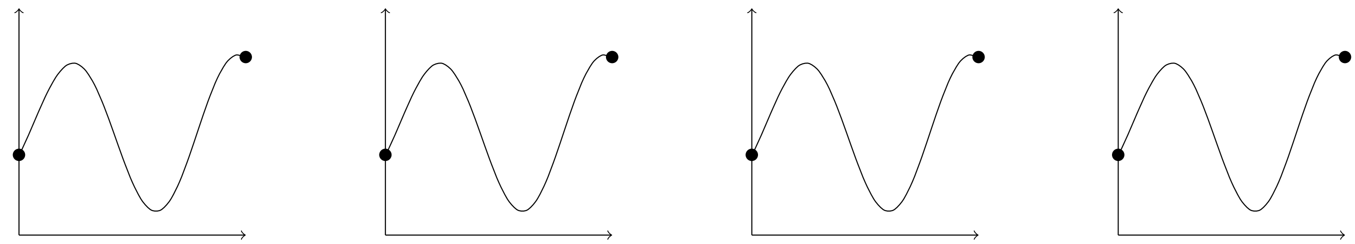
$$L =$$

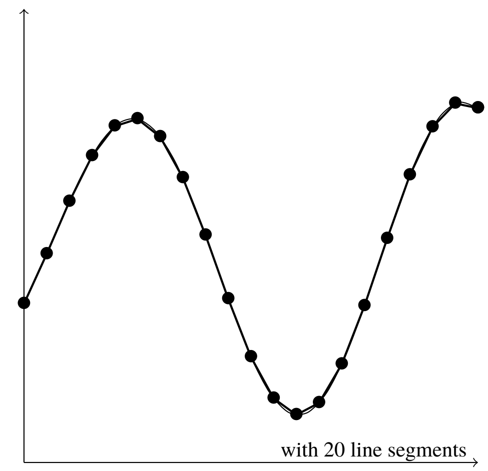
Remark 8.1.2. If a curve has the equation $x = g(y)$ for $c \leq y \leq d$, and $g'(y)$ is continuous then by interchanging the roles of x and y we obtain the following formula for its length:

$$L =$$

Example 8.1.3. Find the arc length for $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ on the interval $1 \leq x \leq 4$.

Idea of a Proof of Theorem 8.1.1





Definition(s) 8.1.4. If a smooth curve C has the equation $y = f(x)$, for $a \leq x \leq b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the ________________________________, and

$$s(x) =$$

Example 8.1.5. Find the arc length function for $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ starting at initial point $x = 1$.

Example 8.1.6. Find the arc length of $f(x) = \ln(\cos x)$ on the interval $0 \leq x \leq \pi/3$.

Example 8.1.7. Setup an integral the represents the arc length function for $f(t) = \tan^{-1}(t)$ with initial point at $t = 1$.

Chapter 11

Infinite Sequences and Series

Forest for the Trees

In Chapter 11 we switch gears to an old topic. Recall back in calculus 1 we found that the tangent line of f near a approximated f near a pretty well (we called it the linear approximation). One natural way to expand on this is ask "What quadratic (2nd degree) polynomial approximates f near a the best?" or how about a 3rd degree polynomial, or 4th, or 5th? Turns out with higher degree polynomials you can get better and better approximations of a function. So what about infinite polynomials? By that I mean polynomials that don't have a highest degree that just go on and on for ever. For instance:

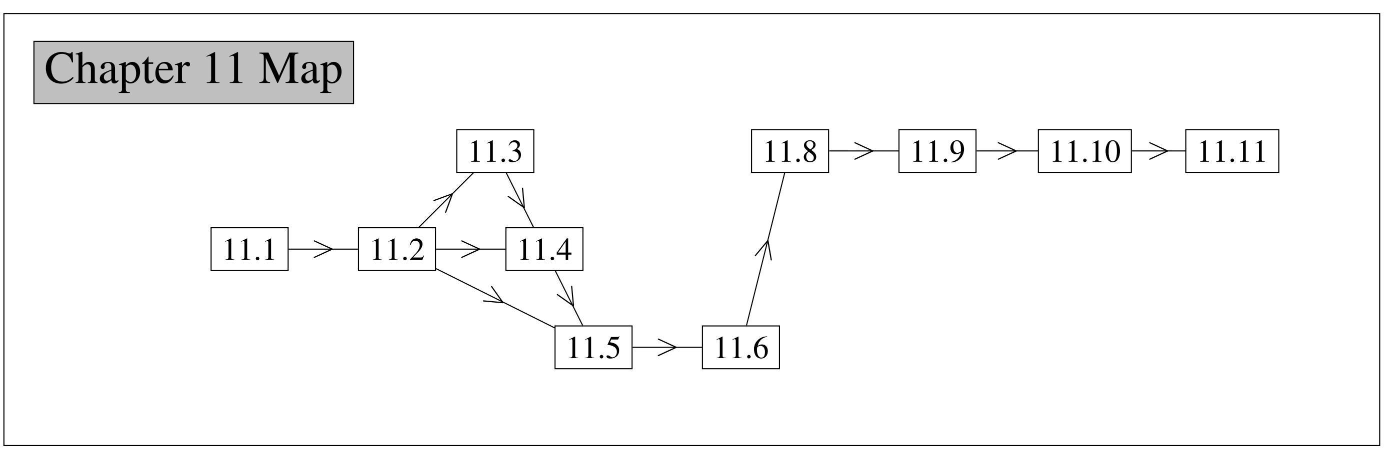
$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

It turns out that sometimes you can take these infinite polynomials (better known as power series) and make them equal to the function itself! So this is our end goal. To get there we need a reminder about sequences and will be practicing with normal series for awhile before upgrading to power series!

By the end of Chapter 11 you should be able to solve problems such as:

Example : Evaluate Find the Maclaurin series for the function $f(x) = \sqrt[3]{1+x}$ and find its radius of convergence.

Example : Evaluate Determine if the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n - 2}{5^{2n}}$ is convergent or divergent. If it is convergent, find what is converges to.



11.1 Sequences

Section Objective(s):

- Examine properties of sequences and their limits
- Use limits of functions to calculate limits of sequences

Definition(s) 11.1.1. A ________________ can be thought of as a list of numbers written in a definite order:

The number ____ is called the *first term*, ____ is the *second term*, and in general ____ is the *nth term*.

Remark 11.1.2. The sequence $\{a_1, a_2, a_3, ...\}$ is also denoted by

Example 11.1.3. Write a function with domain $\mathbb{Z}^+$ that defines the following list of numbers:

12, 14, 16, 18,...

Definition(s) 11.1.4. A sequence $\{a_n\}$ has the ____________ and we write

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ __________,

we say the sequence _________________ (or is ___________________).

Otherwise, we say the sequences _______________ (or is _________________).

Theorem 11.1.5. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then ______________.

Remark 11.1.6. Theorem 11.1.5 implies then that many of the properties of limits of functions apply to limits of sequences!

Theorem 11.1.7. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then

$$\lim_{n\to\infty}(a_n + b_n) = \qquad\qquad \lim_{n\to\infty}(a_n \cdot b_n) =$$

Theorem 11.1.8. Squeeze Theorem for Sequences If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n =$ ___.

Theorem 11.1.9. If $\lim_{n\to\infty} |a_n| =$ __, then $\lim_{n\to\infty} a_n =$ __.

Example 11.1.10. Calculate: $\lim_{n\to\infty} \frac{\cos n}{n}$

Remark 11.1.11. The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n =$$

Definition(s) 11.1.12. A sequence $\{a_n\}$ is called ________________ if ________________ for all $n \geq 1$, that is $a_1 < a_2 < a_3 \cdots$. It is called ________________ if ________________ for all $n \geq 1$. A sequence is ________________ if it is either ________________ or ________________.

Theorem 11.1.13. If $f(x)$ is an ________________ function and $f(n) = a_n$ then $\{a_n\}$ is an ________________ sequence. And similar for ________________.

Example 11.1.14. Show that the following sequence is decreasing: $\left\{\frac{1}{2^n}\right\}$

Example 11.1.15. Consider the sequence: $1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots$. Find a formula for a_n.

Example 11.1.16. Consider the sequence $\left\{\dfrac{1+5n}{3-2n}\right\}$. Find the limit of the sequence.

Example 11.1.17. Consider the sequence $\left\{\left(\dfrac{-1}{5}\right)^n + \dfrac{\sin n}{n}\right\}$. Find the limit of the sequence.

Example 11.1.18. Consider the sequence $\left\{\dfrac{\ln 3n}{2n}\right\}$. Find the limit of the sequence.

11.2 Series

Section Objective(s):

- Define series and some different types of series
- Determine some strategies for identifying when series converge or diverge

Definition(s) 11.2.1. In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

which is called an ________________________ (or just a __________) and is denoted, for short by the symbol

Definition(s) 11.2.2. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called ____________________ and we write

The number s is called the ________ of the series. If the sequence $\{s_n\}$ is divergent, then the series is called _________________.

Example 11.2.3. Determine if the following series converges or diverges: $1 + 2 + 3 + 4 + 5 + \cdots$

Example 11.2.4. Determine if the following series converges or diverges: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Definition(s) 11.2.5. The previous series is called a ____________________________. In general these are of the form

Theorem 11.2.6. The geometric series

is convergent if ____________ and its sum is

If ____________, then the geometric series is divergent.

Theorem 11.2.7. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then ____________________.

Theorem 11.2.8 (nth term test for divergence)**.**
If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ ____________________.

Theorem 11.2.9. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

Example 11.2.10. Determine if the following series is convergent or divergent: $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

Remark 11.2.11. If $\lim_{n\to\infty} a_n = 0$ it does not guarantee that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Definition(s) 11.2.12. The harmonic series is defined as:

Theorem 11.2.13. The harmonic series is divergent

Example 11.2.14. Determine if the following series are convergent or divergent. If they are convergent, find what they converge to

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n - 2}{5^{2n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

11.3 The Integral Test

Section Objective(s):

- Understand the hypotheses and conclusion of the Integral Test.
- Apply the Integral Test to some series

Theorem 11.3.1 (The Integral Test). Suppose $a_n = f(n)$ and that f is

(i)

(ii)

(iii)

on $[1, \infty)$. If all these conditions are met then

(a) If ____________________ is convergent, then ____________ is convergent.

(b) If ____________________ is divergent, then ____________ is divergent.

Remark 11.3.2. Why the Integral Test makes sense

Example 11.3.3. Determine if $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges

Theorem 11.3.4 (p-series test)**.**
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 11.3.5. Determine if the following series is convergent or divergent. $\displaystyle\sum_{i=1}^{\infty} \frac{2}{1+e^i}$

Theorem 11.3.6 (The math of increasing and decreasing positive continuous functions)**.** Suppose $f_1(x)$ and $f_2(x)$ are increasing functions and $g_1(x)$ and $g_2(x)$ are decreasing functions on an interval I then

- $f_1(x) + f_2(x)$ is ____________________
- $f_1(x)f_2(x)$ is ____________________
- $\dfrac{1}{f_1(x)}$ is ____________________
- If $c > 0$ then
  - $cf_1(x)$ is ____________________
  - $-cf_1(x)$ is ____________________
- $g_1(x) + g_2(x)$ is ____________________
- $g_1(x)g_2(x)$ is ____________________
- $\dfrac{1}{g_1(x)}$ is ____________________
- If $c > 0$ then
  - $cg_1(x)$ is ____________________
  - $-cg_1(x)$ is ____________________

Proof of one of these:

Example 11.3.7. Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{-3}{\sqrt{k^3}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{n \ln n}$

11.4 The Comparison Tests

Section Objective(s):

- Go over the Direct and Limit Comparison Tests
- Apply the Comparison tests to a variety of series problems

Theorem 11.4.1 (The Direct Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If _________ is convergent and _____________ for all n, then _________ is also convergent.

(ii) If _________ is divergent and _____________ for all n, then _________ is also divergent.

Theorem 11.4.2 (The Limit Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example 11.4.3. Determine if the following series is convergent or divergent: $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3n-1}$

Example 11.4.4. Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-n-1}$

Example 11.4.5. Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{ne^{-n^2}}{3+e^{-n}}$

11.5 Alternating Series

Section Objective(s):

- Define Alternating Series
- Determine when Alternating Series converge and diverge.

Definition(s) 11.5.1. An ______________________________ is a series whose terms are alternately positive and negative. Heres an example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots =$$

Remark 11.5.2. The nth term of an alternating series is of the form

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

Theorem 11.5.3 (Alternating Series Test). If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

(i)

(ii)

then the series is __________________.

Example 11.5.4. Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\cdot 3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n^2}{1+e^{2n}}$

Example 11.5.5. Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+3}-\sqrt{n})$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\cos\left(\frac{2\pi}{n}\right)$

11.6 Absolute Convergence and the Ratio Test

Section Objective(s):

- Define absolutely and conditionally convergent.
- Understand the statement of the Ratio Test
- Apply the Ratio test to a variety of series

Definition(s) 11.6.1. A series $\sum a_n$ is called ______________________________ if the series of absolute values $\sum |a_n|$ is convergent.

Definition(s) 11.6.2. A series $\sum a_n$ is called ______________________________ if it is convergent but not absolutely convergent.

Remark 11.6.3. The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 11.6.4. If a series $\sum a_n$ is absolutely convergent then it is ______________.

Example 11.6.5. Determine if the following series is absolutely or conditionally convergent: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Theorem 11.6.6. The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is ______________________________ (and therefore ______________).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is ____________.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test ____________________; that is, ___________________ can be drawn about the convergence or divergence of $\sum a_n$.

Example 11.6.7. See what happens when you try to apply the ratio test to

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Remark 11.6.8. Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

Example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Example 11.6.9. Determine if the following series is convergent or divergent: $\sum \frac{(2n)!}{(n!)^2}$

Example 11.6.10. Determine if the following series is absolutely convergent, conditionally convergent, or divergent: $\sum \frac{(-1)^n n^2}{n!}$

Example 11.6.11. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

(a) $\sum n\left(\frac{3}{5}\right)^n$

(b) $\sum \frac{\cos(\pi n)}{3n}$

11.8 Power Series

Section Objective(s):

- Define power series.
- Begin calculating closed forms of power series.
- Calculate radii and interval of convergence for power series.

Definition(s) 11.8.1. A ____________________ is a series of the form

where x is a variable and the c_n's are constants called the coefficients of the series.

So now instead of a series approaching a number we can think that power series approach functions.

Additionally,

Definition(s) 11.8.2. A series of the form

is called a ____________________________________ or a

____ or a ________________________________.

Example 11.8.3. Find a closed expression for the following series: $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\frac{1}{8}(x-2)^3+\cdots$

Theorem 11.8.4. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i)

(ii)

(iii)

Remark 11.8.5. The number R in case (iii) is called the ____ of the power series. By convention, the radius of convergence is ___________ in case (i) and _____________ in case (ii). The ____ of a power series is the interval that consists of all values of x for which the series converges.

Example 11.8.6. Find the open interval of convergence for the following power series: $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

Example 11.8.7. Find the radius and interval of convergence for the following power series: $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{n^2+1}$

Example 11.8.8. Find the radius and interval of convergence for the following power series: $\sum_{n=1}^{\infty} x^n \ln n$

Example 11.8.9. Find the radius and interval of convergence for the following power series: $\displaystyle\sum_{n=0}^{\infty} n!x^n$

Example 11.8.10. Find the interval of convergence for the following power series: $\displaystyle\sum_{n=1}^{\infty}(-1)^n(3x+5)^n$

11.9 Representations of Functions as Power Series

Section Objective(s):

- Understand how functions can be represented as power series
- Learn differentiation and integration for power series

Theorem 11.9.1.

Example 11.9.2. Express $\dfrac{1}{1-3x^2}$ as a power series. Find the radius of the convergence.

Example 11.9.3. Express $\dfrac{x}{3+x}$ as a power series. Find the interval of the convergence.

Theorem 11.9.4. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

is differentiable (and therefore continuous) on the interval ____________________________ and

(i) $f'(x) =$

(ii) $\displaystyle\int f(x)\, dx =$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 11.9.5. Find a power series representation for $\dfrac{1}{(1-x)^2}$ and its radius of convergence.

Example 11.9.6. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Example 11.9.7. Find a power series representation for $\tan^{-1}(1+x)$ and its radius of convergence.

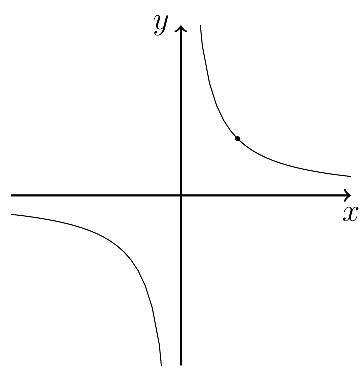
11.10 Taylor and Maclaurin Series

Section Objective(s):

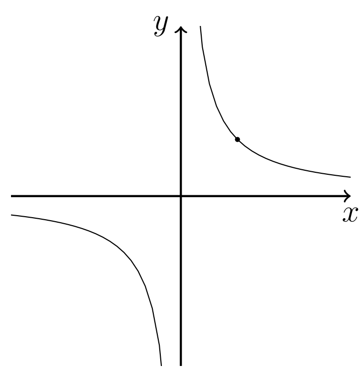
- Define and compute Taylor and Maclaurin Series
- Define and compute Taylor and Maclaurin Polynomials
- Recognize what these polynomials and series represent.
- Define and calculate the remainder

Let's consider the function $f(x) = 1/x$ and polynomials that approximate $f(x)$ as best they can at $a = 1$.

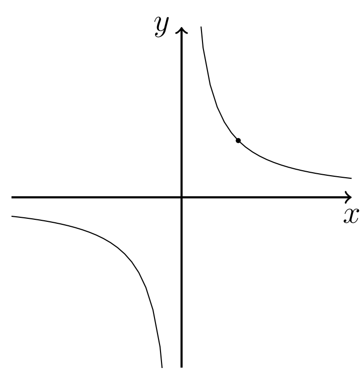
The best degree 0 polynomial (which we will call $T_0(x)$) can agree with f at 1. That is



The best degree 1 polynomial (which we will call $T_1(x)$) can agree with f at 1 and have the same slope. We should remember this from Calculus 1 as the ____________________



The best degree 2 polynomial (which we will call $T_2(x)$) can agree with f at 1, have the same slope, and the same concavity. That is



Theorem 11.10.1. If f has a power series representation (expansion) at a, that is, if

$$f(x) =$$

then its coefficients are given by the formula

$$c_n =$$

Definition(s) 11.10.2. If f has a power series expansion at a, then it must be of the following form:

$$f(x) =$$

This is called the ____ (or **about** a or **centered at** a). For the special case $a = 0$ the Taylor series becomes

$$f(x) =$$

This case arises frequently enough that it is given the special name ______________________.

Example 11.10.3. Find the 2^{nd} degree Maclaurin polynomial for $f(x) = \sqrt{x+1}$.

Definition(s) 11.10.4. In the case of the Taylor series the partial sums are

$T_n(x) =$

Notice that T_n is ____________________ of degree n called the ____. In general, $f(x)$ is the sum of its Taylor series if

$$f(s) = \underline{\hspace{4cm}}$$

Definition(s) 11.10.5. If we let

so that

Then __________ is called the __________________ of the Taylor series.

Example 11.10.6. Find the Taylor series generated by $f(x) = 1/x$ centered at $a = 2$.

Theorem 11.10.7 (Important Maclaurin Series and their Radii of Convergence).

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \qquad R = 1$$

Proof: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and that $R = \infty$

Example 11.10.8. Consider $f(x) = e^{4x}$

(a) Find the n^{th} term Maclaurin polynomial for f using **Definition 11.10.4**.

(b) Find the n^{th} term Maclaurin polynomial for f using **Theorem 11.10.7**.

Theorem 11.10.9. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree polynomial of f at a and

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

Theorem 11.10.10. Taylor Remainder Theorem Suppose $f(x) = T_n(x) + R_n(x)$ where T_n is the nth degree Taylor polynomial of f at a. Then R_n is the Lagrange form of the remainder of order n and is given by

$$R_n(x) =$$

for some c between a and x.

Example 11.10.11. Find the Taylor polynomial of degree 3 for the function $f(x) = \sqrt{x+9}$ about the point $x = -5$. Then write $R_3(x)$ as a function of x and c.

11.11 Additional Applications and Problems for Taylor Polynomials

Section Objective(s):

- Use Taylor polynomials to calculate terms for various binomial series
- Multiply known Taylor polynomials to find the terms of Taylor polynomials for additional functions
- Use Taylor polynomials to help evaluate limits
- Use Taylor polynomials to determine what a series converges to.
- State and apply the Taylor Inequality

Definition(s) 11.11.1. A generalized binomial coefficient can be defined as:

$$\binom{\alpha}{0} = 1$$
$$\binom{\alpha}{1} = \frac{\alpha}{1!} = \alpha$$
$$\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} = \tfrac{1}{2}\alpha(\alpha-1)$$
$$\vdots$$
$$\binom{\alpha}{n} =$$

Example 11.11.2. Find the first four terms of the binomial series for the following function:

$f(x) = (1 + x^3)^{-1/2}$

Example 11.11.3. Find the Maclaurin series for the function $f(x) = \sqrt[3]{1+x}$ and find its radius of convergence.

Example 11.11.4. Find the first 3 terms for the Maclaurin series for the function $f(x) = e^{2x}\sin(x)$

Example 11.11.5. Find the first 4 terms for the Maclaurin series for the function $f(x) = \dfrac{x}{\cos(x)}$

Example 11.11.6. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin(x^3) - x^3}{x^9}$

Example 11.11.7. Evaluate $\lim\limits_{x\to 0} \dfrac{x - \ln(x+1)}{x^2}$

Example 11.11.8. Use Taylor series to evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}$

Example 11.11.9. Use Taylor series to evaluate $\dfrac{1}{1 \cdot 2} - \dfrac{1}{3 \cdot 2^3} + \dfrac{1}{5 \cdot 2^5} - \dfrac{1}{7 \cdot 2^7} + \cdots$

Theorem 11.11.10. Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

Example 11.11.11. Consider approximating $f(x) = e^{2x}$ with $T_2(x) = 1 + 2x + 2x^2$. Find the maximum error by making this approximation on the interval $[-2, 2]$.

Chapter 10

Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric equations

Section Objective(s):

- Define parametric equations.
- Use parametric equations to sketch curves.
- Transform between parametric and Cartesian equations.

Definition(s) 10.1.1. Suppose that x and y are both given as functions of a third variable t (called a ____________________) by the equations

(called ____). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a ________________________________.

Example 10.1.2. Plot the parametric equations $x = 2t$ and $y = 3 - t$. Find the corresponding Cartesian equation

Remark 10.1.3. Sometimes we restrict t to lie in a finite interval. In general, the curve with parametric equations

has ______________________ $(f(a), g(a))$ and ___________________________ $(f(b), g(b))$

Example 10.1.4. Plot the parametric equations $x = 2\sin t$ and $y = 2\cos t$ for $t \in [0, \pi/2]$. Find the corresponding Cartesian equation.

Example 10.1.5. Plot the parametric equations $x = 3\sin 2t$ and $y = -5\cos 2t$ for $t \in [0, \pi/2]$. Find the corresponding Cartesian equation.

Example 10.1.6. Find the corresponding Cartesian equation for the parametric equations $x = 2\sec(3t)$ and $y = -5\tan(3t)$.

Example 10.1.7. Find the a parametric equation for $y = x^2 - x + 3$ which starts at $(0, 3)$ when $t = 0$ and ends a $(3, 9)$ when $t = 5$.

10.2 Calculus with Parametric Curves

Section Objective(s):

- Apply methods of calculus to parametric curves
- Practice solving problems involving tangents and arc length

Definition(s) 10.2.1. Suppose the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$ and that f and g are differentiable functions. Assuming we want to find the tangent line at a point on the curve where y is also a differentiable function of x then the Chain Rule gives

If $dx/dt \neq 0$, we can solve for dy/dx:

Example 10.2.2. Find the equation of the tangent line to the curve given by:

$$x = 3t - \sin(t), \quad y = 1 - \cos(t), \quad \text{at } t = \pi/3$$

Example 10.2.3. Find an equation of the tangent to the curve $x = 1 + \ln t$, $y = t^2 + 3$ at the point $(1, 4)$

Example 10.2.4. Find the points on the curve $x = t^3 - 3t$, $y = t^2 - 3$ where the tangent is horizontal or vertical.

Theorem 10.2.5. If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once at t increases from α to β, then the length of C is

Example 10.2.6. Find the length of a circle of radius 3 defined by: $x = 3\cos(t), y = 3\sin(t)$ on $0 \leq t \leq 2\pi$

Example 10.2.7. Find the exact length of the curve $x = 1 + 3t^2$, $y = 2 + 2t^3$ for $t \in [0, 1]$

Example 10.2.8. Find the length of the curve $x = e^t + e^{-t}$, $y = 1 - 2t$ for $t \in [0, 2]$

10.3 Polar Coordinates

Section Objective(s):

- Understand the concept of polar coordinates
- Transfer back and forth between Cartesian and polar coordinates.
- Graph equations given in polar coordinates.
- Find tangent lines given in polar coordinates.

What are polar coordinates?

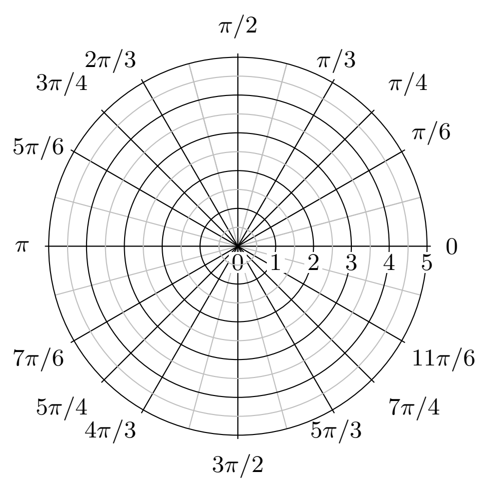
Theorem 10.3.1.

Example 10.3.2. Graph the polar coordinates

(a) $(\theta, r) = (7\pi/6, 2)$

(b) $(\theta, r) = (-\pi/3, -3)$

(c) $(\theta, r) = (3\pi, 0)$



Example 10.3.3. Convert the following polar coordinates into Cartesian coordinates

(a) $(\theta, r) = (0, 2)$

(b) $(\theta, r) = (-\pi/4, 5)$

Example 10.3.4. Convert the following Cartesian coordinates into polar coordinates

(a) $(x, y) = (0, 3)$

(b) $(x, y) = (-2, 2)$

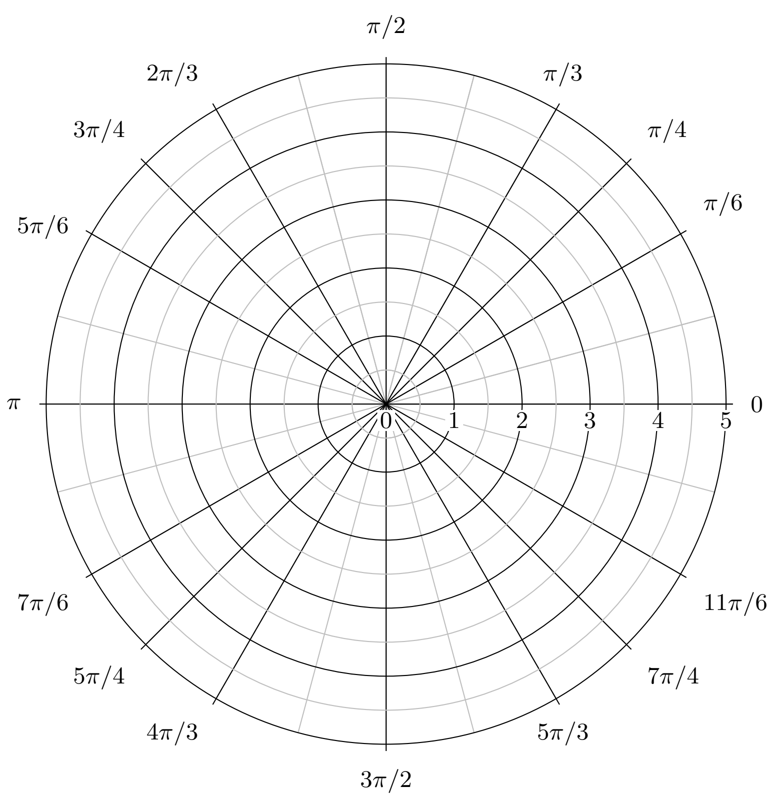
Definition(s) 10.3.5. The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example 10.3.6. Consider the curve given by $r = 2\sin\theta$

(a) Graph the curve using the point plotting method

(b) What geometric figure is the curve?

(c) Transform the curve into Cartesian coordinates to confirm your suspicions in part (b)

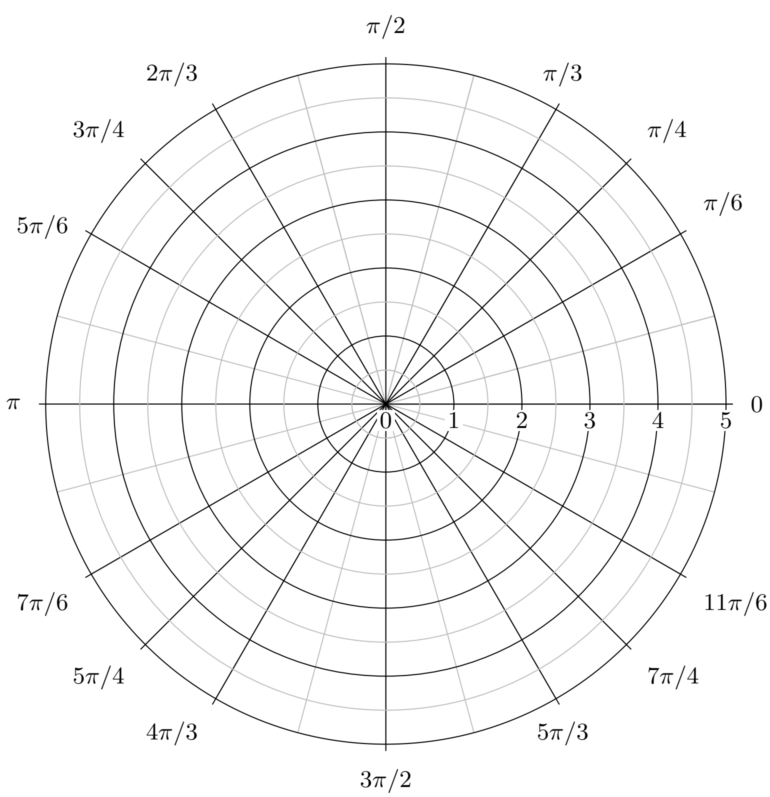


Example 10.3.7. Consider the curve given by $\dfrac{1}{r} = \sin\theta + \cos\theta$ for $\theta \in (-\pi/4, 3\pi/4)$

(a) Graph the curve using the point plotting method

(b) What geometric figure is the curve?

(c) Transform the curve into Cartesian coordinates to confirm your suspicions in part (b)



Definition(s) 10.3.8. To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

Then using the method of finding slopes of parametric curves and the Product Rule, we have

Example 10.3.9. Find the slope of the tangent line to the curve $r = 1 + 2\cos\theta$ at $\theta = \pi/6$.

10.4 Areas and Lengths in Polar Coordinates

Section Objective(s):

- Apply the formula for the area of a region whose boundary is given by polar coordinates
- Apply the formula for the arc length of a curve given by polar coordinates

Theorem 10.4.1. Suppose the boundary of a region R is given by the polar equation $r = f(\theta)$. The area A of R is given by

or

Example 10.4.2. Find the area enclosed by the cardioid $r = 2(1 + \cos(\theta))$.

Remark 10.4.3. Don't forget, symmetry is your friend.

A quick lesson in intersecting functions

Example 10.4.4. Consider the functions $r = 2\cos\theta$ and $r = 2\sin\theta$ for $\theta \in [0, \pi]$.

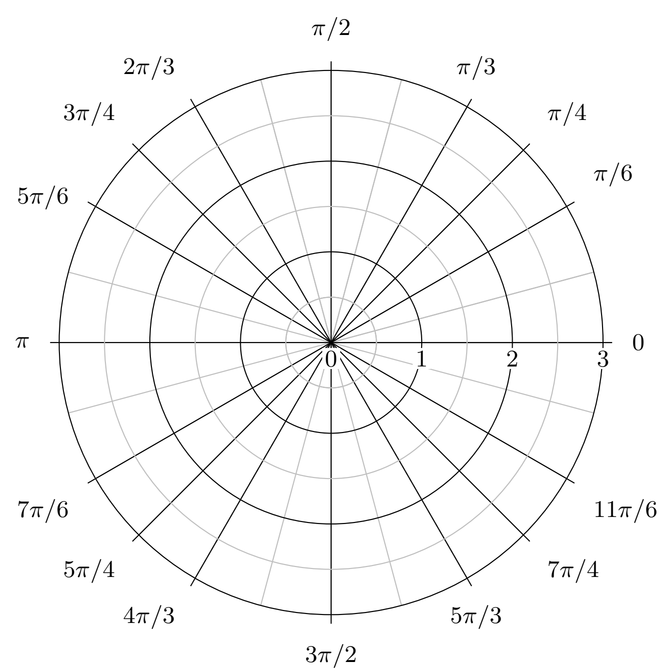
(a) Graph both of the functions together on the graph below.

(b) How would the picture look if you graphed these curves with $\theta \in [0, 2\pi]$?

(c) Find all (r, θ) values were these two functions intersect on.

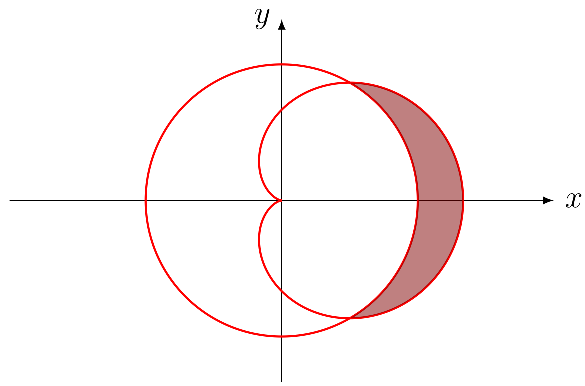
(d) Find all (x, y) values were these two functions intersect on.

(e) Calculate the area shared by these two circles.



Theorem 10.4.5. Suppose $f(\theta)$ and $g(\theta)$ are continuous polar functions and that $f(\theta) \geq g(\theta) \geq 0$ for all $\theta \in [\alpha, \beta]$. The area bounded between $f(\theta)$ and $g(\theta)$ on $[\alpha, \beta]$ is given by

Example 10.4.6. Find the area inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3/2$ (shown below)



Theorem 10.4.7. The length of a curve with polar equation $r = f(\theta), a \leq \theta \leq b$, is

Example 10.4.8. Find the exact length of the polar curve $r = 2\cos\theta$ for $0 \leq \theta \leq \dfrac{3\pi}{4}$

Example 10.4.9. Find the exact length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq \pi$