The Third Annual Herzog Price Examination

November 8, 1975

Problem 1: (Kelly) Prove that $2^{2^n} - 1$ has at least n distinct prime divisors.

Problem 2: (McCoy) Let $y_0 = \sqrt{2}/2$ and recursively define

$$\sqrt{2}\, y_{n+1} = \sqrt{1 + y_n}\,.$$

Compute

$$\lim_{n\to\infty} 4^n(1 - y_n)$$

Problem 3: (D. Wright) In a large field there are thousands of gophers. Each gopher has his own hole and is within one yard of his hole. Suddenly a shot is fired and all gophers bolt for their holes. Show how each gopher may return to its hole staying at all times within 10 yards of his hole and yet never cross the path of any other gopher. (If you cannot do the problem with 10 yards try 1 mile, but remember that the field is very large).

Problem 4: (Sonneborn)

Suppose $a_1, a_2, \cdots, a_n$ is a sequence of integers which takes on at most k distinct values. Prove that if $n \geq 2^k$ then some product of consecutive a_i is a perfect square.

Problem 5: (Sledd) Suppose $f(x)$ is positive, continuously differentiable, and

$$0 < f(x^2) + f'(x)$$

everywhere on $[0,\infty)$. Prove that

$$\int_0^\infty f(x)\,dx$$

diverges.

Problem 6: (Kelly) Prove that it is impossible to partition a rectangular box into a finite number of pairwise incongruent cubes with pairwise disjoint interiors.