THE SEVENTEENTH HERZOG PRIZE EXAMINATION

November 11, 1989

Problem 1

A circle meets each side of a heptagon (7-sided polygon) in exactly one point and does not pass through any vertex. Show that the circle must be tangent to at least one of the sides of the heptagon.

Problem 2

Determine (with proof) for which integers n

$$f(n) = n^5 + n^4 - n - 1$$

is divisible by 15.

Problem 3

Determine (with proof) which is larger, e^{π} or π^e.

Problem 4

Show that the i^{th} root of i is a real number and make a rough estimate of its value. (Here $i = \sqrt{-1}$.)

Problem 5

Let n be a positive integer. Find the value of the determinant $|A|$ of order n with entries

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}.$$

Problem 6

Does the inequality $(2n)^n + (2n+1)^n > (2n+2)^n$ hold for <u>all</u> positive integral values of n? Prove your answer.