THE EIGHTEENTH HERZOG PRIZE EXAMINATION

November 10, 1990

Problem 1

Solve the inequality $|x(1-x)| < 1$ in the real number system, and express its solution set in terms of intervals.

Problem 2

Show that the indeterminate form 0^0 can take on any real value whatsoever.

Problem 3

Prove: If the center of a circle has irrational cartesian coordinates, then there are at most two points on the circle with rational coordinates.

Problem 4

The odd divisors of a given positive integer are listed (for example, the odd divisors of 12 are 1 and 3; those of 45 are 1, 3, 5, 9, 15 and 45). If this experiment is performed many times, it is amazing to some people how often the number of divisors of the form $4k+1$ exceeds that of the divisors of the form $4k+3$. Prove: Every positive integer has at least as many distinct divisors of the form $4k+1$ as of the form $4k+3$.

Problem 5

Ordinary dice have six faces (one with each of the numbers 1,2,3,4,5,6). Determine whether or not it is possible to assign positive integers to each of the faces of two cubes in such a way that (a) neither of the resulting dice is ordinary, and (b) the probability of rolling each of the possible totals 2 through 12 with these dice is the same as rolling the same total with a pair of ordinary dice.

Problem 6

Let $P(x)$ be a polynomial with zeros $r_1,\ldots,r_n$ none of which is repeated, and let $s_1,\ldots,s_{n-1}$ be the zeros of its derivative $P'(x)$.

Evaluate

$$\sum_{i,j} \frac{1}{r_i - s_j} .$$