THE TWENTY-SIXTH HERZOG PRIZE EXAMINATION

November 7, 1998

1. Prove that

$$1998 < \sum_{n=1}^{1,000,000} n^{-1/2} < 1999.$$

2. An even number, say $2n$, of distinct points are chosen on a circle; half of them are labeled "+1" and the other half "-1". A caterpillar equipped with a tiny calculator is assigned to crawl clockwise around the circle, adding up the +1's and –1's as it passes the labeled points. Prove that it is always possible to choose a starting point on the circle so that the total on the caterpillar's calculator is never negative for the entire duration of the creature's circuit.

3. A triangle is situated in the plane in such a way that the rectangular Cartesian coordinates of all three of its vertices are integers. Prove that the tangents of all three of its angles are rational numbers. (For this problem, consider tan $\pi/2$ to be rational.)
Does the same conclusion hold for a triangle in 3-dimensional space?

4. Find, with proof, a closed form expression for $1!1 + 2!2 + 3!3 + \ldots + 1998!1998$.

5. The plane is divided into unit squares, checkerboard fashion, and a positive integer written in each square. The number in each square is the average of the four numbers in the squares that share a side with it. Prove that all the squares have the same number written in them.

6. If you multiply ten consecutive positive integers, you get a large number ending in some zeros. For example, if your first number is 11, then the answer is the product of the integers 11 through 20, which is 670,442,572,800. Notice that the last digit before the zeros is *even*, and more often that not this will be the case. There are some choices of starting numbers, however, that will lead to the last non-zero digit of that long product being odd. Find one. Better yet, if you can, find the *smallest* positive integer k such that the last non-zero digit of $k(k+1)(k+2)\ldots(k+9)$ is odd.